THE FUNDAMENTAL MANTRAS OF COMPACTNESS.

These are the key properties of compact sets that you should internalize. Chant them like mantras as you walk to class in the morning. Know them backwards and forwards. Be able to prove them off the top of your head.

(1) Continuous images of compact sets are compact.
(2) Closed subsets of compact sets are compact.
(3) Compact subsets of Hausdorff spaces are closed.

Let's look at them in more detail. These proofs are a little wordy for clarity. You can probably trim them down a bit.

Theorem 1. *Let $f : K \to Y$ be continuous, where K is compact. Then $f(K)$ is compact in Y.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be a covering of $f(K)$ by open sets U_α. Since f is continuous, every set $f^{-1}(U_\alpha)$ will be open in K. Since the U_α cover $f(K)$, we have an open cover $\{f^{-1}(U_\alpha)\}$ of K. K is compact by hypothesis, so there must be some finite subcover of K; by reindexing the subscripts, we can write it down as $\{f^{-1}(U_i)\}_{i=1}^n$. Then

$$K \subseteq \bigcup_{i=1}^n f^{-1}(U_i) \quad \Longrightarrow \quad f(K) \subseteq f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) \qquad \text{by §1.3 Exercise 2(e).}$$

Now, since

$$\begin{aligned} f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) &= \bigcup_{i=1}^n f\left(f^{-1}(U_i)\right) && \text{by §1.3 Exercise 2(f)} \\ &\subseteq \bigcup_{i=1}^n U_i && \text{by §1.3 Exercise 1(b),} \end{aligned}$$

we have $f(K)$ contained in the finite subcover $\{U_i\}_{i=1}^n$. I.e., $f(K)$ is compact. □

Theorem 2. *Let K be a compact space, and let B be a closed subset of K. Then B is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B. Since B is closed, we know that its complement $\tilde{B}$ is open. Then

$$\{U_\alpha\}_{\alpha \in A} \cup \tilde{B}$$

is an open cover of the whole space K, and hence has a finite open subcover

$$\{U_i\}_{i=1}^n \cup \tilde{B}.$$

Since $\tilde{B}$ doesn't cover any part of B, we can throw it out and still have that

$$\{U_i\}_{i=1}^n$$

is an open cover of B. This is a finite subcover for B, i.e., B is compact. □

Theorem 3. *Let X be a Hausdorff space, and let K be a compact subset of X. Then K is closed.*

Proof. We will show that $\tilde{K}$ is open, so that K is closed. Pick $x \in \tilde{K}$. We must produce an open set U such that $x \in U \subseteq \tilde{K}$, i.e., an open neighbourhood U of x which is disjoint from K.

For each point $y \in K$, use the Hausdorff property to separate x from y, i.e., choose open neighborhoods of U_y of x and V_y of y which are disjoint. Then $\{V_y\}$ is an open cover of K. By the compactness of K, there must be some finite open subcover of K which we can denote by $\{V_i\}_{i=1}^n$. Then by looking at the neighborhoods of x which correspond to these sets V_i, we have $U_i \cap V_i = \varnothing, \forall i = 1, \ldots, n$. Since $\left(\bigcap_{j=1}^n U_j\right) \subseteq U_i, \forall i = 1, \ldots, n$, we have

$$\left(\bigcap_{j=1}^n U_j\right) \cap V_i = \varnothing, \forall i = 1, \ldots, n,$$

and hence also

$$\left(\bigcap_{i=1}^n U_i\right) \cap \bigcup_{i=1}^n V_i = \varnothing.$$

Thus $(\bigcap_{i=1}^n U_i)$ is disjoint from K (since $K \subseteq \bigcup_{i=1}^n V_i$). Moreover, since $(\bigcap_{i=1}^n U_i)$ is a *finite* union of open sets, it is also open. Therefore, we can take $U = \bigcap_{i=1}^n U_i$ as our open neighbourhood of x which is disjoint from K. □

Test your understanding! Using the above ideas, each of the following exercises may be completed with very little work.

Exercises:

1. Let $f : X \to \mathbb{R}$ be continuous, and let $K \subseteq X$ be compact. Then there exist points $m, M \in K$ such that $f(m) \leq f(x) \leq f(M), \forall x \in K$.

   Note: it is crucial to show that m and M are in K, not just in X. (Use an important property of $\mathbb{R}$.)
   m is called a *(global) minimum* of f and M is called a *(global) maximum*. This result is paraphrased by saying "A continuous function on a compact set K attains its max & min".

2. Show that if $f : X \to Y$ is a continuous map, where X is compact and Y is Hausdorff, then f is a closed map.