Design of Engineering Experiments Part 3 – Analysis of Variance (1)

- Comparing more that two factor levels…**the analysis of variance**
  - ANOVA decomposition of total variability
  - Statistical testing & analysis
  - Checking assumptions, model validity
  - Post-ANOVA testing of means
  - Box-Cox method
- **Sample size** determination

What If There Are More Than Two Factor Levels?

- The *t*-test does not directly apply
- There are lots of practical situations where there are either more than two levels of interest, or there are several factors of simultaneous interest
- The **analysis of variance** (ANOVA) is the appropriate analysis "engine" for these types of experiments – Chapter 3, textbook
- The ANOVA was developed by Fisher in the early 1920s, and initially applied to agricultural experiments
- Used extensively today for industrial experiments

An Example (See pg. 61)

- An engineer is interested in investigating the relationship between the RF power setting and the etch rate for this tool. The objective of an experiment like this is to model the relationship between etch rate and RF power, and to specify the power setting that will give a desired target etch rate.
- The response variable is etch rate.
- She is interested in a particular gas (C2F6) and gap (0.80 cm), and wants to test four levels of RF power: 160W, 180W, 200W, and 220W. She decided to test five wafers at each level of RF power.
- The experimenter chooses 4 **levels** of RF power 160W, 180W, 200W, and 220W
- The experiment is **replicated** 5 times – runs made in random order

An Example (See pg. 62)

Table 3-1 Etch Rate Data (in Å/min) from the Plasma Etching Experiment

Power (W)	Observations 1	2	3	4	5	Totals	Averages
160	575	542	530	539	570	2756	551.2
180	565	593	590	579	610	2937	587.4
200	600	651	610	637	629	3127	625.4
220	725	700	715	685	710	3535	707.0

- Does **changing** the power change the mean etch rate?
- Is there an **optimum** level for power?

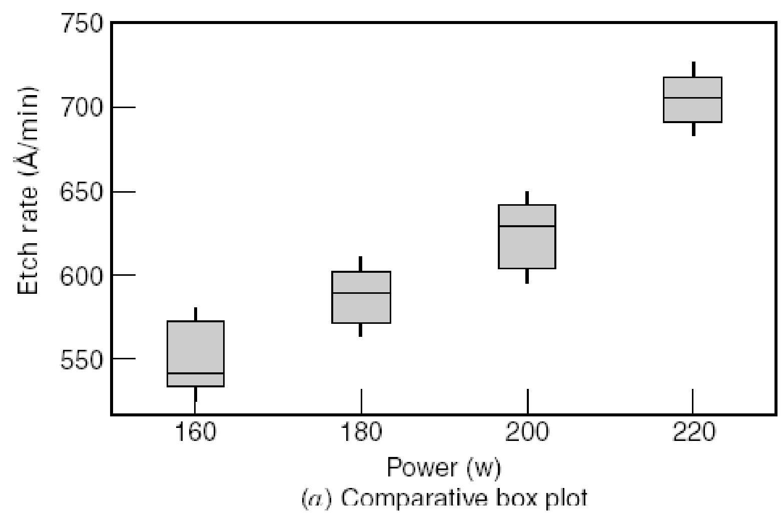

Figure 3-2 Box plots and scatter diagram of the etch rate data.

The Analysis of Variance (Sec. 3-2, pg. 63)

Table 3-2 Typical Data for a Single-Factor Experiment

Treatment (level)	Observations				Totals	Averages
1	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

- In general, there will be a **levels** of the factor, or a **treatments, and** n **replicates** of the experiment, run in **random order…**a completely randomized design **(CRD)**
- $N = an$ total runs
- We consider the **fixed effects** case…the **random effects** case will be discussed later
- Objective is to test hypotheses about the equality of the a treatment means

The Analysis of Variance

- The name "analysis of variance" stems from a **partitioning** of the total variability in the response variable into components that are consistent with a **model** for the experiment
- The basic single-factor ANOVA model is

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \begin{cases} i = 1, 2, ..., a \\ j = 1, 2, ..., n \end{cases}$$

$\mu =$ an overall mean, $\tau_i = ith$ treatment effect,

$\varepsilon_{ij} =$ experimental error, $NID(0, \sigma^2)$

Models for the Data

There are several ways to write a model for the data:

$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ is called the effects model

Let $\mu_i = \mu + \tau_i$, then

$y_{ij} = \mu_i + \varepsilon_{ij}$ is called the means model

Regression models can also be employed

The Hypothesis Testing Framework (Fixed Effect ANOVA)

- Assumption: sampling from a **normal** distribution N(μ_i , σ^2), i=1,2,…a

- Statistical hypotheses:

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$$

$$H_1 : \text{ At least one mean is different}$$

The Analysis of Variance

- **Total variability** is measured by the total sum of squares:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2$$

- The basic ANOVA partitioning is:

$$\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n}[(\bar{y}_{i.} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.})]^2$$

$$= n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2$$

$$SS_T = SS_{Treatments} + SS_E$$

The Analysis of Variance

$$SS_T = SS_{Treatments} + SS_E$$

- A large value of $SS_{Treatments}$ reflects large differences in treatment means
- A small value of $SS_{Treatments}$ likely indicates no differences in treatment means

The Analysis of Variance

- While sums of squares cannot be directly compared to test the hypothesis of equal means, **mean squares** can be compared.
- A mean square is a sum of squares divided by its degrees of freedom:

$$df_{Total} = df_{Treatments} + df_{Error}$$
$$an - 1 = a - 1 + a(n-1)$$

$$MS_{Treatments} = \frac{SS_{Treatments}}{a-1}, MS_E = \frac{SS_E}{a(n-1)}$$

- If the treatment means are equal, the treatment and error mean squares will be (theoretically) equal.
- If treatment means differ, the treatment mean square will be larger than the error mean square.

The Analysis of Variance is Summarized in a Table

Table 3-3 The Analysis of Variance Table for the Single-Factor, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between treatments	$SS_{\text{Treatments}} = n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2$	$a - 1$	$MS_{\text{Treatments}}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Error (within treatments)	$SS_E = SS_T - SS_{\text{Treatments}}$	$N - a$	MS_E	
Total	$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2$	$N - 1$		

- Computing…see text, pp 66-70
- The **reference distribution** for F_0 is the $F_{a-1, a(n-1)}$ distribution
- **Reject** the null hypothesis (equal treatment means) if

$$F_0 > F_{\alpha, a-1, a(n-1)}$$

ANOVA Table
Example 3-1

Table 3-4 ANOVA for the Plasma Etching Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
RF Power	66,870.55	3	22,290.18	$F_0 = 66.80$	<0.01
Error	5339.20	16	333.70		
Total	72,209.75	19			

The Reference Distribution:

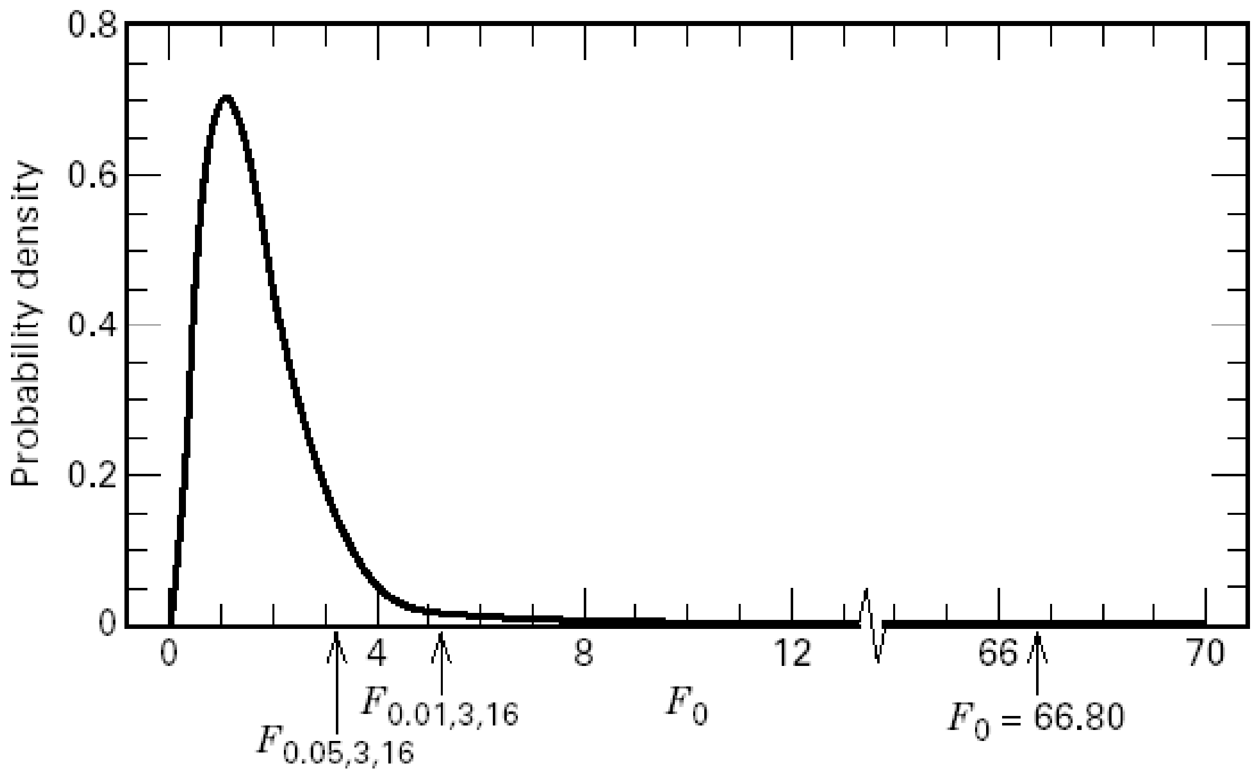

Figure 3-3 The reference distribution ($F_{3,16}$) for the test statistic F_0 in Example 3-1.

ANOVA calculations are usually done via computer

```
##### Read in  data
>plasma=read.table("Plasma.txt", header=T)
>attach(plasma)
#Specify Power is a categorical factor with 4 levels
>Power<-factor(Power)
# 1-way ANOVA
>plasma.aov = aov(EtchRate~Power)
>summary(plasma.aov)

            Df Sum Sq Mean Sq F value    Pr(>F)
Power        3  66871   22290     66.797 2.883e-09 ***
Residuals   16   5339     334
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
1
```

Why Does the ANOVA Work?

We are sampling from normal populations, so

$$\frac{SS_{Treamtents}}{\sigma^2} \sim \chi^2_{a-1} \text{ if } H_0 \text{ is true, and } \frac{SS_E}{\sigma^2} \sim \chi^2_{a(n-1)}$$

Cochran's theorem gives the independence of these two chi-square random variables

$$\text{So } F_0 = \frac{SS_{Treatments}/(a-1)}{SS_E/[a(n-1)]} \sim \frac{\chi^2_{a-1}/(a-1)}{\chi^2_{a(n-1)}/[a(n-1)]} \sim F_{a-1,a(n-1)}$$

$$\text{Finally, } E(MS_{Treatments}) = \sigma^2 + \frac{n\sum_{i=1}^{n}\tau_i^2}{a-1} \text{ and } E(MS_E) = \sigma^2$$

Therefore an upper-tail F test is appropriate.