The Split-Plot Design

- Text reference, Section 14-4 page 540
- The split-plot is a multifactor experiment where it is not possible to completely randomize the order of the runs
- Example – paper manufacturing
  - Three pulp preparation methods
  - Four different temperatures
  - Each replicate requires 12 runs
  - The experimenters want to use three replicates
  - How many batches of pulp are required?
  - The pilot plant is only capable of making 12 runs per day?
  - Pulp preparation methods is a **hard-to-change** factor

The Split-Plot Design

- Consider an alternate experimental design:
  - Run one replicate on each of the three days.
  - In replicate 1(day1), select a pulp preparation method, prepare a batch
  - Divide the batch into four sections or samples, and assign one of the temperature levels to each
  - Repeat for each pulp preparation method
  - Conduct replicates 2 and 3 at day 2 and day 3 similarly

The Split-Plot Design

- Each replicate (sometimes called **blocks**) has been divided into three parts, called the **whole plots**
- Pulp preparation methods is the **whole plot treatment**
- Each whole plot has been divided into four **subplots** or **split-plots**
- Temperature is the **subplot treatment**
- Generally, the hard-to-change factor is assigned to the whole plots
- This design requires only 9 batches of pulp (assuming three replicates)

The Split-Plot Design Model and Statistical Analysis

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{jk}$$

$$+(\tau\beta\gamma)_{ijk} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, a \\ k = 1, 2, \ldots, b \end{cases}$$

There are two error structures; the whole-plot error and the subplot error

Table 14-15 Expected Mean Squares for Split-Plot Design

	Model Term	Expected Mean Square
Whole plot	τ_i	$\sigma^2 + ab\sigma_\tau^2$
	β_j	$\sigma^2 + b\sigma_{\tau\beta}^2 + \frac{rb\sum \beta_j^2}{a-1}$
	$(\tau\beta)_{ij}$	$\sigma^2 + b\sigma_{\tau\beta}^2$
Subplot	γ_k	$\sigma^2 + a\sigma_{\tau\gamma}^2 + \frac{ra\sum \gamma_k^2}{(b-1)}$
	$(\tau\gamma)_{ik}$	$\sigma^2 + a\sigma_{\tau\gamma}^2$
	$(\beta\gamma)_{jk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + \frac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
	$(\tau\beta\gamma)_{ijk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
	$\epsilon_{(ijk)h}$	σ^2 (not estimable)

Split-Plot ANOVA

Table 14-16 Analysis of Variance for the Split-Plot Design Using the Tensile Strength Data from Table 14-14

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Replicates (or Blocks)	77.55	2	38.78		
Preparation method (*A*)	128.39	2	64.20	7.08	0.05
Whole Plot Error (replicates (or Blocks) × *A*)	36.28	4	9.07		
Temperature (*B*)	434.08	3	144.69	41.94	<0.01
Replicates (or Blocks) × *B*	20.67	6	3.45		
AB	75.17	6	12.53	2.96	0.05
Subplot Error (replicates (or Blocks) × *AB*)	50.83	12	4.24		
Total	822.97	35			

Calculations follow a three-factor ANOVA with one replicate

Note the two different **error structures**; whole plot and subplot

```
>sum1=summary(aov(strength~Block*Method*Temp+ Error(Block/Method/Temp)))
> sum1
```

Error: Block

	Df	Sum Sq	Mean Sq
Block	2	77.556	38.778

Error: Block:Method

	Df	Sum Sq	Mean Sq
Method	2	128.389	64.194
Block:Method	4	36.278	9.069

Error: Block:Method:Temp

	Df	Sum Sq	Mean Sq
Temp	3	434.08	144.69
Block:Temp	6	20.67	3.44
Method:Temp	6	75.17	12.53
Block:Method:Temp	12	50.83	4.24

```
AovTab=sum1[[2]][[1]]
(F.method=AovTab[1,3]/AovTab[2,3])
(p.method=1-pf(F.method,AovTab[1,1],AovTab[2,1]))
AovTab=sum1[[3]][[1]]
(F.temperature=AovTab[1,3]/AovTab[2,3])
(p.temperature=1-pf(F.temperature,AovTab[1,1],AovTab[2,1]))
(F.method.temperature=AovTab[3,3]/AovTab[4,3])
(p.method.temperature=1-pf(F.method.temperature,AovTab[3,1],AovTab[4,1]))
```

Alternate Model for the Split-Plot

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\beta\gamma)_{jk} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, a \\ k = 1, 2, \ldots, b \end{cases}$$

Factor	E(MS)
τ_i (Replicates or Blocks)	$\sigma_\epsilon^2 + ab\sigma_\tau^2$
β_j (A)	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2 + \frac{rb\sum \beta_j^2}{a-1}$
$(\tau\beta)_{ij}$	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2$ (whole plot error)
γ_k (B)	$\sigma_\epsilon^2 + \frac{ra\sum \gamma_k^2}{ab-1}$
$(\beta\gamma)_{jk}$ (AB)	$\sigma_\epsilon^2 + \frac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
ϵ_{ijk}	σ_ϵ^2 (subplot error)

```
(sum3=summary(aov(strength~Method*Temp+Error(Block/Method)))

Error: Block
          Df Sum Sq Mean Sq F value Pr(>F)
Residuals  2 77.556  38.778

Error: Block:Method
          Df  Sum Sq Mean Sq F value  Pr(>F)
Method     2 128.389  64.194  7.0781 0.04854 *
Residuals  4  36.278   9.069
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Error: Within
            Df Sum Sq Mean Sq F value    Pr(>F)
Temp         3 434.08  144.69 36.4266 7.449e-08 ***
Method:Temp  6  75.17   12.53  3.1538   0.02711 *
Residuals   18  71.50    3.97
```

Always Remember!!!!

- **carefully consider how the experiment must be conducted and incorporate all restrictions on randomization into the analysis**

- **An example: two factors-visual acuity (A) with levels *a* and illumination level (B) with levels *b*, and n replicates.**
    - **Design 1: all abn observations be taken in random order**
    - **Design 2: adjust the device to one of the *a* acuity and *b* illumination levels and run all n observations at once.**

- Design 1:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \underset{\substack{\downarrow \quad \downarrow \\ \phi_{ijk} \ \theta_{ijk}}}{\varepsilon_{ijk}}$$

ϕ_{ijk} – the scatter or noise in the system that results from "experimental error" (i.e, the failure to duplicate exactly the same levels of acuity and illumination on different runs, etc.)

θ_{ijk} – reproducibility error of the subject

$$E(MS_E) = \sigma_\phi^2 + \sigma_\theta^2$$

- The error mean square has ab(n-1)degrees of freedom
- Standard Two-way ANOVA would work and all the main effects and interaction can be tested.

- Design 2: $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \phi_{ij} + \theta_{\mathrm{ijk}}$

  ϕ_{ijk} is the same for all n values of k

  θ_{ijk} – reproducibility error of the subject

$$E(\mathrm{MS_A}) = \sigma_\theta^2 + n\sigma_\phi^2 + \frac{bn\sum \tau_i^2}{a-1}$$

$$E(\mathrm{MS_B}) = \sigma_\theta^2 + n\sigma_\phi^2 + \frac{an\sum \beta_j^2}{b-1}$$

$$E(\mathrm{MS_{AB}}) = \sigma_\theta^2 + n\sigma_\phi^2 + \frac{n\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$E(\mathrm{MS_E}) = \sigma_\theta^2$$

- The error mean square has ab(n-1)degrees of freedom
- For both factors are fixed as above, there are no tests on the main effects unless interaction is negligible.
  - If both factors are random, then the main effects may be tested against the AB interaction.
  - If only one factor is random, then the fixed factor can be tested against the AB interaction