The Two-Factor Mixed Model

- Two factors, factorial experiment, factor *A* fixed, factor *B* random (Section 12-3, pg. 522)

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

$$V(\beta_j) = \sigma_\beta^2, V[(\tau\beta)_{ij}] = [(a-1)/a]\sigma_{\tau\beta}^2, V(\varepsilon_{ijk}) = \sigma^2$$

$$\sum_{i=1}^{a} \tau_i = 0, \sum_{i=1}^{a} (\tau\beta)_{ij} = 0$$

- The model parameters β_j and ε_{ijk} are *NID* random variables, the interaction effect is normal, but not independent $\operatorname{cov}[(\tau\beta)_{ij}, (\tau\beta)_{i'j}] = -\frac{1}{a}\sigma_{\tau\beta}^2, \mathrm{i} \neq \mathrm{i}'$
- This is called the **restricted model**

Testing Hypotheses - Mixed Model

- Once again, the standard ANOVA partition is appropriate
- Relevant hypotheses:

$$H_0 : \tau_i = 0 \qquad H_0 : \sigma_\beta^2 = 0 \qquad H_0 : \sigma_{\tau\beta}^2 = 0$$

$$H_1 : \tau_i \neq 0 \qquad H_1 : \sigma_\beta^2 > 0 \qquad H_1 : \sigma_{\tau\beta}^2 > 0$$

- Test statistics depend on the **expected mean squares**:

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}$$

$$E(MS_B) = \sigma^2 + an\sigma_\beta^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_E}$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}$$

$$E(MS_E) = \sigma^2$$

Estimating the Variance Components – Two Factor Mixed model

- Use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}_{\beta}^{2} = \frac{MS_B - MS_E}{an}$$

$$\hat{\sigma}_{\tau\beta}^{2} = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^{2} = MS_E$$

- Estimate the fixed effects (treatment means) as usual

Example 13-3 (pg. 497)
The Measurement Systems Capability Study Revisited

- Same experimental setting as in example 13-2
- Parts are a random factor, but Operators are fixed
- Assume the restricted form of the mixed model

```
# Factor A=operator fixed ; Factor B=part random
# Interaction A:B random
> aov1=aov(y~operator*part)
> sum1=summary(aov1)[[1]]
> sum1
                Df  Sum Sq Mean Sq F value Pr(>F)
operator         2    2.62    1.31  1.3193  0.2750
part            19 1185.43   62.39 62.9151 <2e-16 ***
operator:part   38   27.05    0.71  0.7178 0.8614
Residuals       60   59.50    0.99
```

 # The F tests for B and A:B are valid!!!!!

 # The F test for A

```
> F.A=sum1[1,3]/sum1[3,3]
> p.A=1-pf(F.A,sum1[1,1],sum1[3,1])
> c(F.A, p.A)
[1] 1.8379544 0.1730102
```

```
# The variance estimates
> n<-2
> a<-3
> sigma2.E=sum1[4,3] # error variance
> sigma2.B=(sum1[2,3]-sigma2.E)/n/a
> sigma2.AB=(sum1[3,3]-sigma2.E)/n

> c(sigma2.B,sigma2.AB,sigma2.E)
[1] 10.2331871   -0.1399123    0.9916667
```

Example 13-3
Balanced ANOVA

- There is a large effect of parts (not unexpected)
- Small operator effect
- No Part – Operator interaction
- Negative estimate of the Part – Operator interaction variance component
- Fit a reduced model with the Part – Operator interaction deleted
- This leads to the same solution that we found previously for the two-factor random model

Finding Expected Mean Squares

- Obviously important in determining the form of the test statistic
- In fixed models, it's easy:

$$E(MS) = \sigma^2 + f(\text{fixed factor})$$

- Can always use the "brute force" approach – just apply the expectation operator
- Straightforward but tedious
- Rules on page 502-504 work for any balanced model

- Rule 1: the error term in the model is $\varepsilon_{ij\ldots m}$, where the subscript m denotes the replication subscript and the variance component associated with $\varepsilon_{ij\ldots m}$ is σ^2

  example: two-factor model, ε_{ijk}, with associated variance component σ^2

- Rule 2: In addition to an overall mean (μ) and an error term $\varepsilon_{ij\ldots m}$, the model contains all the main effects and any interactions that the experimenter assumes exist. If all possible interactions between k factors exist, then there are $\binom{k}{2}$ two-way interactions, $\binom{k}{3}$,...,1 k-way interaction. If one of the factors in a term appears in parentheses, then there is no interaction between that factor and the other factors in that term.

- Rule 3: For each term in the model other than μ and the error term, divide the subscripts into three classes: (a) live- those subscripts that are present in the term and are not in parentheses; (b) dead-those subscripts that are present in the term and are in parentheses; (c) absent- those subscripts that are present in the model but not in that particular term.

  Example: two-factor fixed effects model: no dead subscripts, i,j are live, k is absent

- Rule 4: the number of degrees of freedom for any effect in the model is the product of the number of levels associated with each dead subscript and the number of levels minus 1 associated with each live subscript. The number of degrees of freedom for error is obtained by subtracting the sum of all other degrees of freedom from N-1, where N is the total number of observations

  Example: two-factor model: $(\tau\beta)_{ij}$ d.f. (a-1)(b-1)

- Rule 5: Each term in the model has either a variance component (random effect) or a fixed factor (fixed effect) associated with it. If an interaction contains at least one random effect, the entire interaction is considered as random. A variance component has Greek letters as subscripts to identify the particular random effect. A fixed effect is always represented by the sum of squares of the model components associated with that factor divided by its degrees of freedom

Example: two-factor mixed model with A fixed and B random

The variance component for B is σ_β^2 and the variance component for AB is $\sigma_{\tau\beta}^2$, the fixed effect for A is

$$\frac{\sum_{i=1}^{a} \tau_i^2}{a-1}$$

- Rule 6: There is an expected mean square for each model component. The expected mean square for error is $E(MS_E)=\sigma^2$. In the case of the restricted model, for every other model term, the expected mean square contains either the variance component or the fixed effect component for that term, plus those components for all other model terms that contain the effect in question and that involve no interactions with other fixed effects, plus σ^2. The coefficient of each variance component of fixed effect is the number of observations at each distinct value of that component.

Example 1: two-factor fixed effects model,

$$E(MS_{AB}) = \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)} + \sigma^2$$

$$E(MS_A) = \frac{bn\sum_{i=1}^{a}\tau_i^2}{(a-1)} + \sigma^2$$, factor B is fixed effect in the interaction term

Example 2: two-factor random effects model

$$E(MS_{AB}) = n\sigma_{\tau\beta}^2 + \sigma^2$$

$$E(MS_A) = bn\sigma_\tau^2 + n\sigma_{\tau\beta}^2 + \sigma^2$$, factor B is random effect in the interaction term

Example 3: two-factor mixed effects model

$$E(MS_{AB}) = n\sigma_{\tau\beta}^2 + \sigma^2$$

$$E(MS_A) = bn\frac{\sum_{i=1}^{a}\tau_i^2}{a-1} + n\sigma_{\tau\beta}^2 + \sigma^2$$, factor B is random effect in the interaction term

$$E(MS_B) = an\sigma_\beta^2 + \sigma^2$$, factor A is fixed effect in the interaction term

Approximate *F* Tests

- Sometimes we find that there are no exact tests for certain effects (page 505)
- Leads to an approximate *F* test (“pseudo” *F* test)
- Test procedure is due to Satterthwaite (1946), and uses **linear combinations** of the original mean squares to form the *F*-ratio
- The linear combinations of the original mean squares are sometimes called “synthetic” mean squares
- Adjustments are required to the degrees of freedom

Satterthwaite Approximate F- test

- Let

$MS' = MS_r + \ldots + MS_s$ and $MS'' = MS_u + \ldots + MS_v$

- $F = MS'/MS'' \sim F(p,q)$

$$p = \frac{(MS_r + \ldots + MS_s)^2}{MS_r^2 / f_r + \ldots + MS_s^2 / f_s}$$

and

$$q = \frac{(MS_u + \ldots + MS_v)^2}{MS_u^2 / f_u + \ldots + MS_v^2 / f_v}$$

Table 13-11 Analysis of Variance for the Pressure Drop Data

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Squares	Mean Square	F_0	P-Value
Temperature, A	1023.36	2	$\sigma^2 + bn\sigma^2_{\tau\gamma} + cn\sigma^2_{\tau\beta} + n\sigma^2_{\tau\beta\gamma} + \frac{bcn\Sigma\tau_i^2}{a-1}$	511.68	2.22	0.17
Operator, B	423.82	3	$\sigma^2 + an\sigma^2_{\beta\gamma} + acn\sigma^2_{\beta}$	141.27	4.05	0.07
Pressure gauge, C	7.19	2	$\sigma^2 + an\sigma^2_{\beta\gamma} + abn\sigma^2_{\gamma}$	3.60	0.10	0.90
AB	1211.97	6	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$	202.00	14.59	<0.01
AC	137.89	4	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$	34.47	2.49	0.10
BC	209.47	6	$\sigma^2 + an\sigma^2_{\beta\gamma}$	34.91	1.63	0.17
ABC	166.11	12	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$	13.84	0.65	0.79
Error	770.50	36	σ^2	21.40		
Total	3950.32	71				

$$MS' = MS_A + MS_{ABC} = 511.68 + 13.84 = 525.52$$

$$MS'' = MS_{AC} + MS_{AB} = 34.47 + 202.0 = 236.47$$

$$E(MS') - E(MS'') = \frac{bcn\sum \tau_i^2}{a-1}$$

$$F = \frac{MS'}{MS''} = \frac{525.52}{236.47} = 2.22$$

$$p = \frac{(MS_A + MS_{ABC})^2}{(MS_A)^2/2 + (MS_{ABC})^2/12} = \frac{525.52^2}{(511.68)^2/2 + (13.84)^2/12} = 2.11 \approx 2$$

$$q = \frac{(MS_{AC} + MS_{AB})^2}{(MS_{AC})^2/4 + (MS_{AB})^2/6} = \frac{236.47^2}{(34.47)^2/4 + (202)^2/6} = 7.88 \approx 8$$

$$F_{0.05}(2,8) = 4.46$$

Do not reject the null hypothesis

```
#####obtain ANOVA table #######
aov1=aov(y~Temp*Operator*Pressure)

sum1=summary(aov1)[[1]]

> sum1
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
Temp	2	1023.36	511.68	23.9072	2.476e-07	***
Operator	3	423.82	141.27	6.6007	0.001141	**
Pressure	2	7.19	3.60	0.1681	0.845952	
Temp:Operator	6	1211.97	202.00	9.4378	3.108e-06	***
Temp:Pressure	4	137.89	34.47	1.6106	0.192749	
Operator:Pressure	6	209.47	34.91	1.6312	**0.166923**	
Temp:Operator:Pressure	12	166.11	13.84	0.6468	**0.788202**	
Residuals	36	770.50	21.40			

The F tests for interaction ABC, BC are valid

#The F test for interaction AB and AC

```
> F.AB.AC=sum1[4:5,3]/sum1[7,3] ##F-value for AB and AC respectively

> p.AB.AC=1-pf(F.AB.AC,sum1[4:5,1],sum1[7,1])
  #p-value for AB and AC respectively

> c(F.AB.AC,p.AB.AC)
[1] 1.459231e+01 2.490301e+00 6.907075e-05 9.905299e-02
```

	F-value	p-value
AB	14.59	<0.01
AC	2.49	0.099

```
> #The F tests for B and C

> F.B.C=sum1[2:3,3]/sum1[6,3] ##F-value for B and C
respectively

>p.B.C=1-pf(F.B.C,sum1[2:3,1],sum1[6,1])
###p-value for Band C respectively

> c(F.B.C,p.B.C)
[1] 4.04654555 0.10303673 0.06858372 0.90365575
```

	F-value	p-value
B	4.05	0.07
C	0.10	0.90

##The F tests for A

```
> MS1=sum1[1,3]+sum1[7,3]
> MS2=sum1[4,3]+sum1[5,3]

> F.A=MS1/MS2
> p=MS1^2/(sum1[1,3]^2/sum1[1,1]+sum1[7,3]^2/sum1[7,1])
> q=MS2^2/(sum1[4,3]^2/sum1[4,1]+sum1[5,3]^2/sum1[5,1])

> p.A=1-pf(F.A,p,q)

> c(F.A, p.A)
[1] 2.2223897 0.1708498
```

	Fvalue	p-value
A	2.22	0.17