Extension to Factorial Treatment Structure

- Two factors, factorial experiment, both factors random (Section 13-2, pg. 490)

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

$$V(\tau_i) = \sigma_\tau^2, V(\beta_j) = \sigma_\beta^2, V[(\tau\beta)_{ij}] = \sigma_{\tau\beta}^2, V(\varepsilon_{ijk}) = \sigma^2$$

$$V(y_{ijk}) = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$$

- The model parameters are *all normally distributed, independent* random variables with mean 0 and variances given above
- Random effects model

Testing Hypotheses - Random Effects Model

- Once again, the standard ANOVA partition is appropriate
- Relevant hypotheses:

$$H_0 : \sigma_\tau^2 = 0 \qquad H_0 : \sigma_\beta^2 = 0 \qquad H_0 : \sigma_{\tau\beta}^2 = 0$$

$$H_1 : \sigma_\tau^2 > 0 \qquad H_1 : \sigma_\beta^2 > 0 \qquad H_1 : \sigma_{\tau\beta}^2 > 0$$

- Form of the test statistics depend on the **expected mean squares**:

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + bn\sigma_\tau^2 \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}$$

$$E(MS_B) = \sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_\beta^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_{AB}}$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}$$

$$E(MS_E) = \sigma^2$$

Recall EMS for two-way fixed effect

- Expected mean squares:

$$\mathrm{E}(MSE) = \sigma^2,$$

$$\mathrm{E}(MSA) = \sigma^2 + nb\frac{\sum_{i=1}^{a}\alpha_i^2}{a-1}, \quad \mathrm{E}(MSB) = \sigma^2 + na\frac{\sum_{j=1}^{b}\beta_j^2}{b-1},$$

$$\mathrm{E}(MSAB) = \sigma^2 + n\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2}{(a-1)(b-1)}.$$

Estimating the Variance Components – Two Factor Random model

- As before, use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_{AB}}{bn}$$

$$\hat{\sigma}_{\beta}^2 = \frac{MS_B - MS_{AB}}{an}$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^2 = MS_E$$

- Potential problems with these estimators

Example 13-2 (pg. 492)
A Measurement Systems Capability Study

- Gauge capability (or R&R) is of interest
- The gauge is used by an operator to measure a critical dimension on a part
- This is a two-factor factorial (completely randomized) with both factors (operators, parts) **random – a random effects model**

Table 13-3 The Measurement Systems Capability Experiment in Example 13-2

Part Number	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
4	27	27	28	26	27	28
5	19	18	19	18	18	21
6	23	21	24	21	23	22
7	22	21	22	24	22	20
8	19	17	18	20	19	18
9	24	23	25	23	24	24
10	25	23	26	25	24	25
11	21	20	20	20	21	20
12	18	19	17	19	18	19
13	23	25	25	25	25	25
14	24	24	23	25	24	25
15	29	30	30	28	31	30
16	26	26	25	26	25	27
17	20	20	19	20	20	20
18	19	21	19	19	21	23
19	25	26	25	24	25	25
20	19	19	18	17	19	17

```
######obtain ANOVA table ######
aov1=aov(y~part*operator)
sum1=summary(aov1)[[1]]
sum1
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
part	19	1185.43	62.39	62.9151	<2e-16	***
operator	2	2.62	1.31	1.3193	0.2750	
part:operator	**38**	**27.05**	**0.71**	**0.7178**	**0.8614**	
Residuals	60	59.50	0.99			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Only the F test for interaction is valid !!!!!

```
# The F test for  A=part and B=operator ######
###sum[1:2,3] are the values for MSA and MSB#####
###sum1[3,3] are the value for MSE######

F.A.B=sum1[1:2,3]/sum1[3,3] ##F-value for A and B respectively
p.A.B=1-pf(F.A.B,sum1[1:2,1],sum1[3,1]) ###p-value for A and B respectively
```

```
> F.A.B
[1] 87.646950  1.837954

> p.A.B
[1] 0.0000000 0.1730102
```

- **No significant part-operator interaction**
- **Effect of parts is large**
- **Operators may have a small negligible effect**

```
# The variance estimates
sigma2.E=sum1[4,3]
a=sum1[1,1]+1
b=sum1[2,1]+1
n=sum1[4,1]/a/b+1   #df for SSE is ab(n-1)
sigma2.AB=(sum1[3,3]-sigma2.E)/n
sigma2.A=(sum1[1,3]-sum1[3,3])/b/n
sigma2.B=(sum1[2,3]-sum1[3,3])/a/n
c(sigma2.A,sigma2.B,sigma2.AB,sigma2.E)
```

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_{AB}}{bn}$$

$$\hat{\sigma}_{\beta}^2 = \frac{MS_B - MS_{AB}}{an}$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^2 = MS_E$$

```
> c(sigma2.A,sigma2.B,sigma2.AB,sigma2.E)
[1] 10.27982456  0.01491228 -0.13991228  0.99166667
```

Not reasonable. But unavoidable if use ANOVA method of estimation➔set to 0 (reasonable in this case) since interaction is not significant

+

Additional variability in the measurement system resulting from use of the instrument by the operator

Variation observed when the same part is measured by the same operator

Reduced Model

```
aov2=aov(y~part+operator)
sum2=summary(aov2)[[1]]

sum2
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
part	19	1185.43	62.39	70.6447	<2e-16	***
operator	2	2.62	1.31	1.4814	0.2324	
Residuals	98	86.55	0.88			

The two F tests are valid !!!!!

```
# The variance estimates
a=sum2[1,1]+1
b=sum2[2,1]+1
n=(sum2[3,1]+a+b-1)/a/b
sigma2.E=sum2[3,3]
sigma2.A.B=(sum2[1:2,3]-sum2[3,3])/n/c(b,a)
```

```
c(sigma2.A.B,sigma2.E)
[1] 10.25127103  0.01062925  0.88316327
```

Additional variability in the measurement system resulting from use of the instrument by the operator

Variation observed when the same part is measured by the same operator