I. The Hölder Inequality

Hölder: $\|fg\|_1 \leq \|f\|_p \|g\|_q$ for $\frac{1}{p} + \frac{1}{q} = 1$.

What does it give us?
Hölder: $(L^p)^* = L^q$ (Riesz Rep), also: relations between L^p spaces

I.1. **How to prove Hölder inequality.**

(1) Prove Young's Inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

(2) Then put $A = \|f\|_p$, $B = \|g\|_q$. Note: $A, B \neq 0$ or else trivial. Then let $a = \frac{|f(x)|}{A}, b = \frac{|g(x)|}{B}$ and apply Young's:

$$ab = \frac{|f(x)g(x)|}{AB} \leq \frac{|f(x)|^p}{pA^p} + \frac{|g(x)|^q}{qA^q} = \frac{a^p}{p} + \frac{b^q}{q}$$

$$\frac{1}{AB}\int |f(x)g(x)|d\mu \leq \frac{1}{pA^p}\int |f|^p d\mu + \frac{1}{qB^q}\int |g|^q d\mu$$

but $A^p = \int |f|^p d\mu$ and $B^q = \int |g|^q d\mu$, so this is

$$\frac{1}{\|f\|_p\|g\|_q}\|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

I.1.1. *How to prove Young's inequality.*

There are many ways.

1. Use Math 9A. [Lapidus]
   Wlog, let $a, b < \infty$ (otherwise, trivial).
   Define $f(x) = \frac{x^p}{p} + \frac{1}{q} - x$ on $[0, \infty)$ and use the first derivative test:

$$f'(x) = x^{p-1} - 1, \text{ so } f'(x) = 0 \iff x^{p-1} = 1 \iff x = 1.$$

   So f attains its min on $[0, \infty)$ at $x = 1$. $(f'' \geq 0)$.
   Note $f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$ (conj exp!).

$$\text{So } f(x) \geq f(1) = 0 \implies \frac{x^p}{p} + \frac{1}{q} - x \geq 0$$

$$\implies \frac{x^p}{p} + \frac{1}{q} \geq x$$

Ansatz: $x = ab^{1/(1-p)}$. Then $x^p = a^p b^{p/(1-p)} = a^p b^{-q}$:

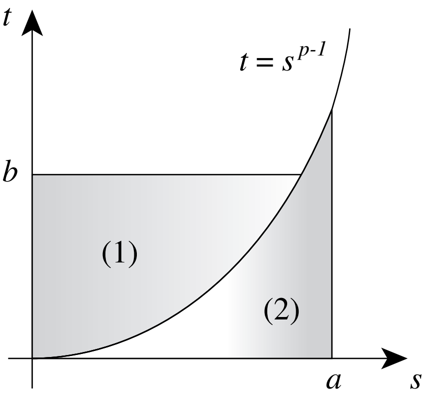
$$ab^{1/(1-p)} \leq \frac{a^p b^{-q}}{p} + \frac{1}{q} \qquad \frac{p}{1-p} = -q$$

$$ab \leq \frac{a^p}{p}\left(b^{\frac{p}{1-p}}\right)\left(b^{-\frac{p}{1-p}}\right) + \frac{1}{q}b^{-\frac{p}{1-p}} \qquad b^{1-\frac{1}{1-p}} = b^{-\frac{p}{1-p}}$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

2. Use Math 9B. [Cohn]
Consider the graph of $t = s^{p-1}$:

FIGURE 1. The graph of $t = s^{p-1}$



Since

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q} \implies p = qq - 1 \implies p - 1 = \frac{1}{q-1},$$

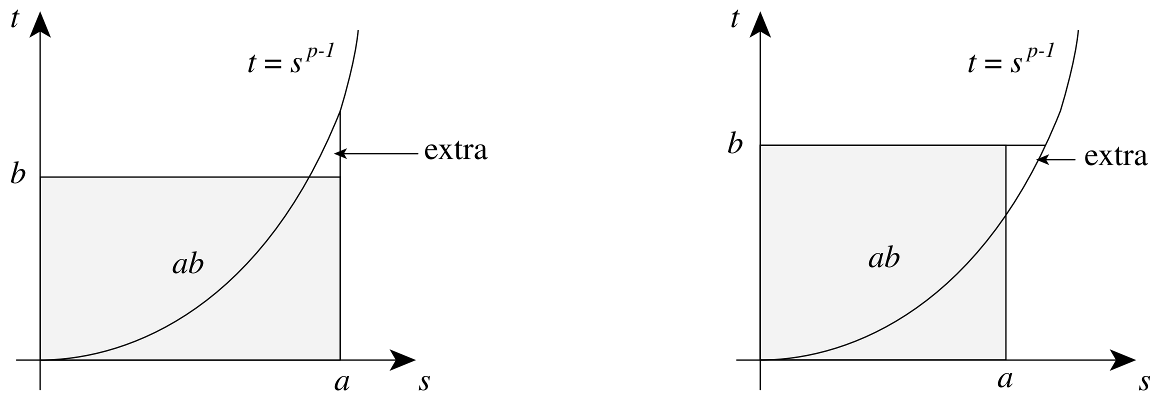
this is also the graph of $s = t^{q-1}$.
Now $(1) = \int_0^a s^{p-1} = \frac{s^p}{p}\Big]_0^a = \frac{a^p}{p}$,
and $(2) = \int_0^b t^{q-1} = \frac{t^q}{q}\Big]_0^b = \frac{b^q}{q}$.

Thus the area of the entire shaded region is $(1)+(2) = \frac{a^p}{p} + \frac{b^q}{q}$, which is clearly always larger than the box of area ab:

I.1.2. *A proof without Young's inequality.*
Use convexity [Rudin]:

$$\varphi\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y).$$

FIGURE 2. The inherent inequality



Since $f \in L^p, g \in L^q$, we have $0 < \|f\|_p, \|g\|_q < \infty$, wlog.
Define $F(x) = \frac{|f(x)|}{\|f\|_p}$ and $G(x) = \frac{|g(x)|}{\|g\|_q}$ so that

$$\int F^p d\mu = \int \frac{|f(x)|^p}{\|f\|_p^p} d\mu = \frac{\int |f|^p d\mu}{\int |f|^p d\mu} = 1$$

and $\int G^q = 1$ similarly.
Now define

$$s(x) = \log \left(\frac{|f(x)|}{\|f\|_p} \right)^p, t(x) = \log \left(\frac{|g(x)|}{\|g\|_q} \right)^q,$$

so that

$$F(x) = e^{s(x)/p} \text{ and } G(x) = e^{t(x)/q}.$$

Since e^x is a convex function, put $\lambda = \frac{1}{q}$ and get

$$e^{\frac{s(x)}{p} + \frac{t(x)}{q}} \leq \tfrac{1}{p} e^{\frac{s(x)}{p}} + \tfrac{1}{q} e^{\frac{t(x)}{q}}$$

$$\implies F(x)G(x) \leq \tfrac{F(x)^p}{p} + \tfrac{G(x)^q}{q}.$$

Now integrate the left side and get

$$\begin{aligned} \|FG\|_1 &= \int |FG| d\mu \\ &= \int \frac{|fg|}{\|f\|_p \|g\|_q} d\mu \\ &= \frac{1}{\|f\|_p \|g\|_q} \int |fg| d\mu \\ &= \frac{\|fg\|_1}{\|f\|_p \|g\|_q}, \end{aligned}$$

and we integrate the right side to get

$$\begin{aligned} \int \left(\frac{F(x)^p}{p} + \frac{G(x)^q}{q} \right) d\mu &= \tfrac{1}{p} \int F^p d\mu + \tfrac{1}{q} \int G^q d\mu \\ &= \tfrac{1}{p} + \tfrac{1}{q} \\ &= 1 \end{aligned}$$

Thus,

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Advantage of this method? No need for Young's!

I.1.3. *Recap - 3 good ways to prove a functional inequality.*
To prove $a(x) \leq b(x)$:

1. Use basic calculus on a difference function:
   Define $f(x) := a(x) - b(x)$.
   Use calculus to show $f(x) \leq 0$ (by computing f', etc)

2. Use geometry.

3. Exploit another inequality. E.g., for any convex function $\varphi(x)$,
   $$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y).$$
   Candidates for φ: $e^x, x^p, \ldots$.

I.1.4. *What did we not do yet? case $p = 1, \infty$.*

$$\begin{aligned} |g(x)| &\leqslant_{\text{ae}} \|g\|_\infty \\ |f(x)| \cdot |g(x)| &\leqslant_{\text{ae}} |f(x)| \cdot \|g\|_\infty \\ |f(x)g(x)| &\leqslant_{\text{ae}} |f(x)| \cdot \|g\|_\infty \\ \|f(x)g(x)\|_1 &\leqslant_{\text{ae}} |f(x)| \cdot \|g\|_\infty \end{aligned}$$

$p = \infty$ is exactly the same.

I.2. **How to use the Hölder inequality.** Assume $(X, \mathcal{M}, \mu)$ is a measure space with $\mu X \geq 1$, $f : X \to \mathbb{R}$ is measure, and $L^p = L^p(X, \mu)$.

1. For $1 \leq p \leq q < \infty$, if $|f(x)| \geq 1$, then $\|f\|_p \leq \|f\|_q$. If $\mu X = \infty$, then $\int_X |f|^p d\mu = \int_X |g|^q d\mu = \infty$, so let $\mu X < \infty$.
   Then
   $$|f|^p \leq |f|^q$$
   $$\int_X |f|^p d\mu \leq \int_X |f|^q d\mu. \tag{I.1}$$
   If $\int_X |f|^q d\mu = \infty$, it is trivial, so assume not.
   Then $\int_X |f|^q d\mu < \infty \implies f \in L^q$, and (I.1) $\implies f \in L^p$.
   Now $p = q \implies \|f\|_p = \|f\|_q$ and we are done trivially, so let $p < q$. We would like to use Hölder with $g(x) = 1$ and some conjugate exponents

α, β with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.
Ansatz: Let $\alpha=\frac{q}{p}$ and $\beta=\frac{q}{q-p}$, so

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{p}{q}+\frac{q-p}{q}=\frac{q}{q}=1.$$

Now use Hölder with $f=f^p$ to get

$$\|f^p g\|_1 \le \|f^p\|_\alpha \|g\|_\beta. \tag{I.2}$$

Now remembering that $g=1$, we have

$$\begin{aligned}
\|f^p g\|_1 &= \|f^p\|_1 = \int_X |f|^p = \|f\|_p^p, \text{ and}\\
\|f^p\|_\alpha &= \left(\int_X |f^p|^{q/p}\right)^{p/q} = \left(\int_X |f|^q\right)^{p/q} = \|f\|_q^p, \text{ and } (*)\\
\|g\|_\beta &= \left(\int_X 1^\beta\right)^{1/\beta} = \left(\int_X 1\right)^{1/\beta} = (\mu X)^{1/\beta}.
\end{aligned}$$

So (I.2) becomes

$$\begin{aligned}
\|f\|_p^p &\le \|f\|_q^p \cdot (\mu X)^{\frac{q-p}{q}}\\
\|f\|_p &\le \|f\|_q \cdot (\mu X)^{(q-p)/pq}\\
\|f\|_p &\le \|f\|_q
\end{aligned}$$

2. For $1 \le p \le q < \infty$, $|f(x)| \le 1\, \forall x \in X \implies \|f\|_p \ge \|f\|_q^{q/p}$.
$p=q$ is trivial, so take $p<q$.
Then

$$\begin{aligned}
p<q, |f| \le 1 \implies |f|^p &\ge |f|^q\\
\int |f|^p &\ge \int |f|^q\\
\left(\int |f|^p\right)^{1/p} &\ge \left(\int |f|^q\right)^{1/p}\\
\|f\|_p &\ge \|f\|_q^{q/p}.
\end{aligned}$$

3. Show $p \le q \le r$ and $f \in L^p, f \in L^r \implies f \in L^q$.
Let $A = \{|f| \ge 1\}$ and $B = \{|f| < 1\} = \widetilde{A}$.

$$\begin{aligned} f \in L^p &\implies \int_X |f|^p = \int_A |f|^p + \int_B |f|^p < \infty \\ &\implies \int_B |f|^p < \infty \qquad \text{(I.3)} \\ f \in L^r &\implies \int_X |f|^r = \int_A |f|^r + \int_B |f|^r < \infty \\ &\implies \int_A |f|^r < \infty. \qquad \text{(I.4)} \end{aligned}$$

On A, $|f|^q \le |f|^r \implies \int_A |f|^q \le \int_A |f|^r$.
On B, $|f|^q \le |f|^p \implies \int_B |f|^q \le \int_B |f|^p$.
So

$$\begin{aligned} \int_X |f|^q &= \int_A |f|^q + \int_B |f|^q \\ &\le \int_A |f|^r + \int_B |f|^p \\ &< \infty \qquad \text{by (I.3),(I.4)} \end{aligned}$$

shows that $f \in L^q$.
Moral: to show $\int_X f(x) \le \int_X g(x)$, try splitting X.

4. Show there is a bounded linear operator $\varphi : L^q \to (L^p)^*$ given by

$$\varphi_f(g) = \varphi(f)(g) = \int_X fg d\mu, \forall f \in L^q, \forall g \in L^p$$

so that $\varphi : g \mapsto \int fg d\mu$ is the functional "integration against f".

$$\begin{aligned}
\|\varphi\| &= \sup_{f\in L^q, f\neq 0} \frac{\|\varphi(f)\|}{\|f\|_q} \\
&= \sup_{f\in L^q, f\neq 0} \sup_{g\in L^p, g\neq 0} \frac{\|\varphi(f)(g)\|}{\|f\|_q\|g\|_p} && \text{def of } \|\varphi\| \\
&= \sup_{f\in L^q, f\neq 0} \sup_{g\in L^p, g\neq 0} \frac{|\int_X fg\, d\mu|}{\|f\|_q\|g\|_p} && \text{def of } \varphi(f)(g) \\
&\le \sup_{f\in L^q, f\neq 0} \sup_{g\in L^p, g\neq 0} \frac{\|fg\|_1}{\|f\|_q\|g\|_p} && |\int fg| \le \int |fg| \\
&\le \sup_{f\in L^q, f\neq 0} \sup_{g\in L^p, g\neq 0} \frac{\|f\|_q\|g\|_p}{\|f\|_q\|g\|_p} && \text{Hölder}
\end{aligned}$$

so $\|\varphi\| \le 1$ and φ is bounded.
To see φ is linear, let $f_1, f_2, f \in L^q, g \in L^p$, and $\alpha \in \mathbb{R}$: we show two things in $(L^p)^*$ are equal by showing that they act the same way on any $g \in L^p$.

$$\begin{aligned}
\varphi(f_1+f_2)(g) &= \int (f_1+f_2)\, g\, d\mu \\
&= \int f_1 g\, d\mu + \int f_2 g\, d\mu \\
&= \varphi(f_1)(g) + \varphi(f_2)(g) = (\varphi(f_1)+\varphi(f_2))\,(g)
\end{aligned}$$

shows $\varphi(f_1+f_2) = \varphi(f_1) + \varphi(f_2)$, and

$$\varphi(\alpha f)(g) = \int \alpha f g\, d\mu = \alpha \int fg\, d\mu = \alpha\varphi(f)(g)$$

shows $\varphi(\alpha f) = \alpha\varphi(f)$.
Hence (by the linearity of the integral), φ is linear.

II. THE DUAL OF L^p

Proposition II.1. Show that $\varphi : L^q \to (L^p)^*$ by $\varphi : f \mapsto \int fg d\mu$ is an isometry.

Proof. So we must show $\|\varphi(f)\| = \|f\|, \forall f \in L^q$. Let $1 < p, q < \infty$. Then

$$\begin{aligned}
\|\varphi(f)\| &= \sup_{g \in L^p, g\neq 0} \frac{\|\varphi(f)(g)\|}{\|g\|_p} && \text{def of } \|\varphi\| \\
&= \sup_{g \in L^p, g\neq 0} \frac{|\int_X fg\, d\mu|}{\|g\|_p} && \text{def of } \varphi(f)(g) \\
&\leq \sup_{g \in L^p, g\neq 0} \frac{\|fg\|_1}{\|g\|_p} && \left|\int fg\right| \leq \int |fg| \\
&\leq \sup_{g \in L^p, g\neq 0} \|f\|_q && \text{Hölder}
\end{aligned}$$

Hence $\|\varphi(f)\| \leq \|f\|$. For $\|\varphi(f)\| \geq \|f\|$, use the fact that $\|\varphi(f)\|$ is defined as a supremum: $\|\varphi(f)\|$ is the smallest number such that

$$\|\varphi(f)(g)\| \leq \|\varphi(f)\| \cdot \|g\| \quad \text{holds } \textit{for all } g\, (\neq 0).$$

In other words, if we can find a g for which $\frac{\|\varphi(f)(g)\|}{\|g\|} \geq \|f\|$, then

$$\|\varphi(f)\| = \sup_{g \in L^p, g\neq 0} \left\{ \frac{\|\varphi(f)(g)\|}{\|g\|} \right\} \geq \|f\|.$$

Ansatz: let $g = |f|^{q/p} \operatorname{sgn} f$.

Then $|g|^p = |f|^q = fg$.[1] Thus, $f \in L^q \implies g \in L^p$. Now

$$\begin{aligned}
\int |g|^p &= \int |f|^q \\
\left(\int |g|^p\right)^{1/p} &= \left(\int |f|^q\right)^{1/p} \\
\left(\int |f|^q\right)^{1/q} \left(\int |g|^p\right)^{1/p} &= \left(\int |f|^q\right)^{1/q} \left(\int |f|^q\right)^{1/p} \\
\|f\|_q \|g\|_p &= \left(\int |f|^q\right)^{1/q+1/p} = \left(\int |f|^q\right)^1 = \|f\|_q^q
\end{aligned}$$

[1] $\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{q}{p} + \frac{q}{q} = q \implies \frac{q}{p} + 1 = q$, so $fg = f \cdot |f|^{q/p} \operatorname{sgn} f = |f| \cdot |f|^{q/p} = |f|^{1+q/p} = |f|^q$.

Thus, $\varphi(f)(g) = \int fg\,d\mu = \int |f|^q = \|f\|_q^q = \|f\|_q \|g\|_p$
$\implies \frac{|\varphi(f)(g)|}{\|g\|_p} \geq \|f\|_q$.

Now suppose $p = 1, q = \infty$.
We have

$$\|\varphi(f)\| = \sup \frac{|\int fg d\mu|}{\|g\|_1} \leq \sup \frac{\|g\|_1 \cdot \|f\|_\infty}{\|g\|_1} = \|f\|_\infty$$

as before. Now it remains to find a $g \in L^1$ for which $\int fg\,d\mu \geq \left(\int |g| d\mu\right) \|f\|_\infty$. We have $f \in L^\infty$, so note $\|f\|_\infty < \infty$. Then fix $\varepsilon > 0$ and define

$$B = \{f \geq \|f\|_\infty - \varepsilon\},$$

and let A be any measurable subset of B such that $0 < \mu A < \infty$.[2] Define

$$g_\varepsilon := \chi_A \operatorname{sgn} f.^3$$

Then $\int |g_\varepsilon| d\mu = \mu A$ and

$$\int f g_\varepsilon\,d\mu = \int_A |f| d\mu \geq (\|f\|_\infty - \varepsilon)\,\mu A$$

$$\frac{\int |f g_\varepsilon| d\mu}{\mu A} \geq \|f\|_\infty - \varepsilon$$

$$\sup \left\{ \frac{\int |f g_\varepsilon| d\mu}{\int |g_\varepsilon| d\mu} \right\} \geq \|f\|_\infty - \varepsilon.$$

Since this is true for any ε, let $\varepsilon \to 0$ and obtain $\sup \left\{ \frac{\int |fg| d\mu}{\int |g| d\mu} \right\} \geq \|f\|_\infty$.

Now suppose $p = \infty, q = 1$.
Again, $\|\varphi(f)\| = \sup \frac{|\int fg d\mu|}{\|g\|_\infty} \leq \|f\|_1$, as before. Now find $g \in L^\infty$ such that

$$\int fg\,d\mu \geq \|g\|_\infty \left(\int |f| d\mu \right).$$

Let $\alpha > 0$ and define $g = \alpha \operatorname{sgn} f$ be a constant function. Then

$$\int fg\,d\mu = \alpha \int |f| d\mu = \|g\|_\infty \int |f| d\mu.$$

□

[2] Note that if f is a constant function, μB could be ∞! $\mu B > 0$ by def of $\|f\|_\infty$.
[3] Include the $\operatorname{sgn} f$ so that $f g_\varepsilon = |f|$ instead of just f, in the following derivation.

II.1. The Riesz Representation Theorem.

Theorem II.2. Let F be a bounded linear functional on L^p, $1 \leq p < \infty$. Then $\exists g \in L^q$ such that $F(f) = \int fg\, d\mu, \forall f \in L^p$, and $\|F\| = \|g\|_q$.

There are two common proofs for this theorem. One uses step functions and absolute continuity of functions; the other uses simple functions and absolute continuity of measures. Both follow a similar strategy:

(1) Show $F(\chi_A) = \int g\chi_A\, d\mu = \int_A g\, d\mu$.
- use absolute continuity of $\Phi : [0,1] \to \mathbb{R}$ by $gF(s) = F(\chi_s)$; or
- use absolute continuity of $\nu E = F(\chi_E)$.

(2) Extend to a dense subspace of L^p
- use step functions for $gF(s)$
- use simple functions for νE

(3) Establish $\|F\| = \|g\|_q$
- extend F to bounded measurable functions, use Royden & Hölder
- define $G(f) = \int fg d\mu$ on L^p and use density, continuity

(4) Extend to L^p
- approx by step functions
- use G

(5) Show uniqueness of g.

Method I:
- uses step functions– only applies for $L^p([a,b],\lambda)$
- requires reference to 3 thms of Royden
- nice use of DCT, boundedness
- absolute continuity of a function is a bit more concrete

Method II:
- uses simple functions, so applies to $L^p(X)$
- requires reference to 1 thm of Royden
- uses Radon-Nikodym Theorem
- smooth use of general topology
- works for any σ-finite μ

Important note: must add "μ is σ-finite" in order to do the case $p = 1$!

Method I (Royden)

Proof.

1. For $s \in [0,1]$, let $\chi_s := \chi_{[0,s]}$. Then $F(\chi_s) = \int_0^s g\,d\lambda$ is some real number, so define $\Phi : [0,1] \to \mathbb{R}$ by
$$\Phi(s) = F(\chi_s) = \int_0^s g\,d\lambda.$$
Claim: Φ is absolutely continuous.

   Fix $\varepsilon > 0$ and let $\{(a_i, b_i)\}_{i=1}^n$ be any finite collection of disjoint subintervals of $[0,1]$ such that
$$\sum (b_i - a_i) < \delta.$$
Then $\sum |\Phi(b_i) - \Phi(a_i)| = F(f)$ for
$$f = \sum_{i=1}^n (\chi_{b_i} - \chi_{a_i}) \operatorname{sgn}(\Phi(b_i) - \Phi(a_i)).$$[4]
Since $\int |f|^p = \sum \int \chi^p_{(a_i,b_i)} = \sum \int_{a_i}^{b_i} 1\,d\lambda = \sum(b_i - a_i) < \delta$,[5]
$$\sum |\Phi(b_i) - \Phi(a_i)| = F(f) \le \|F\| \cdot \|f\|_p < \|F\|\delta^{1/p}.$$
Thus, total variation of Φ over any finite collection of disjoint intervals is less than ε, as long as the total length of these intervals is less than
$$\delta = \tfrac{\varepsilon^p}{\|F\|^p},$$
which shows that Φ is absolutely continuous.

   Then Φ has an antiderivative, by some theorem: [6]
$$\Phi(s) = \int_0^s g.$$

[4]Note that $|\Phi(b_i) - \Phi(a_i)| = |F(\chi_{a_i}) - F(\chi b_i)| = (F(\chi_{a_i}) - F(\chi_{b_i})) \operatorname{sgn}(\Phi(\chi_{b_i}) - \Phi(\chi_{a_i}))$.

[5]In the first equality, use $(\chi_{b_i} - \chi_{a_i}) = \chi_{(a_i,b_i)}$ and $|\operatorname{sgn} h| = 1$ ae.

[6]Royden, Lemma 5.14 on page 110.

Thus $F(\chi_s) = \int_0^1 g\chi_s$.

2. Since every step function φ on $[0,1]$ is a linear combination

$$\varphi = \sum c_i \chi_{s_i} \quad (\lambda\text{-ae}),$$

we get

$$F(f) = \int_0^1 g\varphi$$

by the linearity of F, $\int$.

3. Now extend F to the bounded measurable functions f on $[0,1]$. Let f be such a function, and find a *bounded* sequence $\{\varphi_n\} \subseteq \mathcal{S}tep$ which converges λ-ae to f. This is possible by Royden, Prop 3.22. Then $\{|f-\varphi_n|^p\}$ is uniformly bounded ($\exists M$ such that $|f-\varphi_n|^p \leq M^p$, $\forall n, x$) and tends to 0, λ-ae.
This bound allows us to use the DCT and get

$$\lim_{n\to\infty} \int |f-\varphi_n|^p = 0 \quad \Longrightarrow \quad \|f-\varphi_n\|_p \xrightarrow{n\to\infty} 0.$$

Then the boundedness of F and the fact that

$$|F(f) - F(\varphi_n)| = |F(f-\varphi_n)| \leq \|F\| \cdot \|f-\varphi_n\|$$

together imply that $F(f) = \lim_{n\to\infty} F(\varphi_n)$.
Also, $|g\varphi_n| \leq |g| \cdot M_\varphi \implies \int fg = \lim \int g\varphi_n$ by the DCT again.[7]
Putting this together,

$$F(f) = \lim F(\varphi_n) = \lim \int g\varphi_n = \int fg, \quad \forall f \text{ (bdd,measurable)}.$$

Then by Proposition 5.12 on page 131 of Royden,

$$\left| \int fg\, d\lambda \right| = |F(f)| \leq \|F\| \cdot \|f\|_p N \cdot \|f\|_p$$

and we have $g \in L^q$, $\|g\|_q \leq N = \|F\|,$.
Then by Prop II.1, $\|F\| = \|g\|_q$.

[7] M_φ is the uniform bound on the sequence $\{\varphi_n\}$.

4. Extend to $f \in L^p$.
$\forall \varepsilon, \exists \psi \in \mathcal{S}tep$ such that $\|f - \psi\|_p < \varepsilon$.
Then $\psi \in \mathcal{S}tep \implies \psi$ bounded $\implies F(f) = \int \psi g$, by (3).
Hence,

$$\begin{aligned}\left|F(f) - \int fg\right| &= \left|F(f) - F(\psi) + \int \psi f - \int fg\right| \\ &\leq |F(f) - F(\psi)| + \left|\int fg - \int \psi f\right| \\ &= |F(f - \psi)| + \left|\int (g - \psi) f\right| \\ &\leq \|F\| \cdot \|f - \psi\|_p + \|f - \psi\|_p \cdot \|g\|_q \\ &< (\|F\| + \|g\|_q)\, \varepsilon\end{aligned}$$

Since ε is arbitrary, this shows $F(f) = \int fg\, d\lambda$.

5. If g_1, g_2 determine the same F in this way, then

$$\int fg_1\, d\lambda - \int fg_2\, d\lambda = \int f(g_1 - g_2)\, d\lambda$$

gives the zero functional, and

$$\|g_1 - g_2\|_q = 0 \qquad \implies \qquad g_1 \underset{\text{ae}}{=} g_2.$$

$\square$

Method II (Royden)
Again, given a bounded linear functional F on L^p, we must find a $g \in L^q$ such that $F(f) = \int fg\, d\mu$.

Proof. First, consider a finite measure space $(X, \mathcal{A}, \mu)$.
Then f bounded $\implies f \in L^p(\mu)$.

1. Define ν on $\mathcal{A}$ by

$$\nu E = F(\chi_E).$$

Then ν is a signed measure:
If E is the disjoint union of $\{E_n\} \subseteq \mathcal{A}$, let $\alpha_n = \operatorname{sgn} F(\chi_{E_n})$ and define $f = \sum \alpha_n \chi_{E_n}$. Then F is bounded, so a lemma gives

$$\sum_{n=1}^{\infty} |\nu E_n| = F(f) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \nu E_n = F(\chi_E) = \nu E.$$

So ν is a signed measure. Also,

$$\mu = 0 \implies F(\chi_E) = 0,$$

so $\nu \ll \mu$. Then Radon-Nikodym applies and $\exists g$ measure such that $\nu E = \int_E g\, d\mu$.
Note: F bounded $\implies F(\chi_X) = \nu X < \infty$, so ν is finite.
Then $\nu X = \int_X g\, d\mu < \infty \implies g \in L^1(\mu)$.

2. Let φ be a simple function. Then by linearity of F and $\int$, we have

$$F(\varphi) = \int \varphi g\, d\mu.$$

Now $|F(\varphi)| \leq \|F\| \cdot \|\varphi\|_p \implies g \in L^q$ by some Lemma on page 283.

3. (&4.) Define $G(f) = \int fg\, d\mu$ for $f \in L^p$, so $(G - F)$ is a BLT on L^p which vanishes on $\mathcal{S}$.

$$(G - F) \text{ bounded} \equiv (G - F) \text{ continuous},$$

so $(G - F) = 0$ on L^p.
Hence, $\forall f \in L^p, F(f) = \int fg\, d\mu$ and $\|F\| = \|G\| = \|g\|_q$.

5. If g_1, g_2 determine the same F in this way, then

$$\int fg_1\, d\mu - \int fg_2\, d\mu = \int f(g_1 - g_2)\, d\mu$$

gives the zero functional, and

$$\|g_1 - g_2\|_q = 0 \qquad \implies \qquad g_1 \underset{\text{ae}}{=} g_2.$$

Now consider μ σ-finite. Choose $\{X_n\}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n, \quad X_n \subseteq X_{n+1}, \quad \mu X_n < \infty, \forall n.$$

Then the previous case gives a g_n for each X_n such that g_n vanishes outside X_n and $F(f) = \int f g_n \, d\mu$ for all $f \in L^p$ that vanish outside X_n.

By construction, the uniqueness of g_n on X_n gives

$$g_{n+1}\Big|_{X_n} = g_n,$$

so for $x \in X$, define

$$g(x) := g_n(x), \text{ where } x \in X_n.$$

Since g_n differs from g_m on a set of at most measure 0 (on any X_i where both are defined), discrepancies may be safely ignored and g is well-defined. Moreover, $|g_n|$ increases pointwise to $|g|$. By MCT,

$$\int |g|^q d\mu = \lim \int |g_n|^q d\mu \leq \|F\|^q,$$

so $g \in L^q$.

For general $f \in L^p$, define

$$f_n = \begin{cases} f & \text{on } X_n \\ 0 & \text{on } \widetilde{X_n} \end{cases}.$$

Then $f_n \xrightarrow{\text{pw}} f$ and $f_n \xrightarrow{L^p} f$.
Then $|f_n g| \leq |fg| \in L^1$, so DCT gives

$$\begin{aligned} \int fg \, d\mu &\overset{\text{DCT}}{=} \lim \int f_n g \, d\mu \\ &\overset{\text{DCT}}{=} \lim \int f_n g_n \, d\mu \\ &= \lim F(f_n) \\ &= F(f). \end{aligned}$$

□

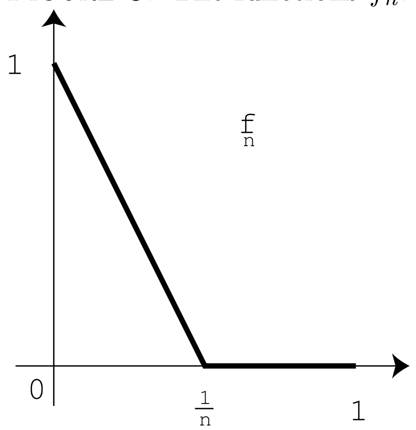
Side note: if λ is Lebesgue measure and ν is the point mass at 0 (on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$), then what *would* $\frac{d\nu}{d\mu}$ be (if $\nu \ll \mu$)?

II.2. $(L^1)^* = L^\infty$, but $(L^\infty)^* \neq L^1$.

Let X be $[0,1]$ so that we can safely consider $C(X)$ as a subspace of $L^\infty(X)$. Define $\zeta : C(X) \to \mathbb{R}$ by $\zeta(f) = f(0)$, so $\zeta \in (C(X))^*$.

By HBT, $\exists \varphi \in (L^\infty)^*$ such that $\varphi(f) = f(0) \, \forall f \in C(X)$. To see that φ cannot be given by integration against a function in L^1, consider $f_n \subseteq C(X)$ defined by $f_n(x) = \max\{1 - nx, 0\}$.

FIGURE 3. The functions f_n.



Then $\varphi(f_n) = f_n(0) = 1 \, \forall n$.
But $f_n(x) \to 0 \, \forall x > 0$, so $f_n g \to 0 \, \forall g \in L^1$.
If $\varphi(f_n) = \int f_n g$, then we would have

$$\begin{aligned} 1 = \varphi(f) &= \varphi(\lim f_n) \\ &= \lim \varphi(f_n) && \varphi \in (L^\infty)^* \implies \varphi \text{ continuous} \\ &= \lim \int f_n g && \text{hypothesis} \\ &= \int \lim f_n g && \text{DCT} \\ &= \int 0 \\ &= 0 \end{aligned}$$

Slightly fancier version: use $\zeta(f) = f(p)$ and $f_n(x) = \max\{0, 1 - n|x - p|\}$.

III. THE MINKOWSKI INEQUALITY

$(X, \mathcal{M}, \mu)$, $1 \le p \le \infty$. Then $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

What does it give us? L^p is a normed vector space ($\triangle$-ineq), L^p is Banach.

III.1. How to prove Minkowski inequality.

III.1.1. *Use Hölder inequality.*

case i) $1 < p < \infty$.

For $f, g \in L^p$, define q by $\frac{1}{p} + \frac{1}{q} = 1$, so that $p + q = pq$:

$$\begin{aligned} \tfrac{1}{q} &= 1 - \tfrac{1}{p} = \tfrac{p-1}{p} \\ \implies q &= \tfrac{p}{p-1} \\ \implies p + q &= \tfrac{p^2-p}{p-1} + \tfrac{p}{p-1} = \tfrac{p^2}{p-1} = p\left(\tfrac{p}{p-1}\right) = pq \end{aligned}$$

Now

$$\left(|f+g|^{p-1}\right)^q = \left(|f+g|^{p-1}\right)^{p/(p-1)} = |f+g|^p.$$

Since L^p is a vector space, we have $f+g \in L^p$ and hence $|f+g|^{p-1} \in L^q$. We need to set up for Hölder:

$$\begin{aligned} |f+g|^p &= |f+g|^{p-1} \cdot |f+g| \\ &\le |f+g|^{p-1}\left(|f|+|g|\right) \\ &\le |f+g|^{p-1}|f| + |f+g|^{p-1}|g| \end{aligned}$$

$$\int |f+g|^p \, d\mu \le \int |f+g|^{p-1}|f| \, d\mu + \int |f+g|^{p-1}|g| \, d\mu$$

Now Hölder gives

$$\int |f| \cdot |f+g|^{p-1} \, d\mu \le \|f\|_p \| |f+g|^{p-1} \|_q \quad \text{and}$$

$$\int |g| \cdot |f+g|^{p-1} \, d\mu \le \|g\|_p \| |f+g|^{p-1} \|_q.$$

Thus

$$\int |f+g|^p \, d\mu \le (\|f\|_p + \|g\|_p) \, \||f+g|^{p-1}\|_q$$
$$= (\|f\|_p + \|g\|_p) \left(\int |f+g|^{(p-1)q} d\mu \right)^{1/q}.$$

If $\int |f+g|^p d\mu = 0$, then Minkowski is trivial, so assume not. Then we may divide by

$$\left(\int \left(|f+g|^{p-1}\right)^q d\mu \right)^{1/q} = \left(\int |f+g|^p d\mu \right)^{1/q}$$

to get

$$\left(\int |f+g|^p d\mu \right)^{1-1/q} \le \|f\|_p + \|g\|_p.$$

case ii) $p = 1$.

Then

$$|f+g| \le |f| + |g| \qquad \triangle\text{-ineq}$$
$$\int |f+g| d\mu \le \int |f| d\mu + \int |g| d\mu \qquad \text{integration}$$
$$\|f+g\|_1 \le \|f\|_1 + \|g\|_1$$

case iii) $p = \infty$.

Then define the null sets

$$N_1 := \{|f(x)| > \|f\|_\infty\}, \qquad N_2 := \{|g(x)| > \|g\|_\infty\}.$$

Then $f, g \in L^\infty \implies \mu N_1 = \mu N_2 = 0$, so $\mu(N_1 \cup N_2) = 0$ also. On the complement of $N_1 \cup N_2$ we have

$$|f(x) + g(x)| \le |f(x)| + |g(x)|.$$

Then taking suprema gives

$$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

III.1.2. *Use convexity.*
Let $\alpha = \|f\|_p$ and $\beta = \|g\|_p$. Note: $\alpha, \beta \neq 0$ or else trivial. Then define

$$f_0 := \tfrac{1}{\alpha}|f|, \quad g_0 := \tfrac{1}{\beta}|g|$$

so that these functions satisfy

$$|f| = \alpha f_0, \quad |g| = \beta g_0, \quad \|f_0\|_p = \|g_0\|_p = 1.$$

Note that this implies

$$\|f_0\|_p^p = \|g_0\|_p^p = 1 \tag{III.1}$$

Set

$$\lambda := \tfrac{\alpha}{\alpha+\beta} \quad \text{so } 1 - \lambda = \tfrac{\alpha+\beta}{\alpha+\beta} - \tfrac{\alpha}{\alpha+\beta} = \tfrac{\beta}{\alpha+\beta}.$$

Then we have

$$\begin{aligned}
|f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p && \triangle\text{-ineq} \\
&= (\alpha f_0(x) + \beta g_0(x))^p && \text{def of } f_0, g_0 \\
&= (\alpha+\beta)^p \left(\tfrac{\alpha}{\alpha+\beta} f_0(x) + \tfrac{\beta}{\alpha+\beta} g_0(x)\right)^p \\
&= (\alpha+\beta)^p \left(\lambda f_0(x) + (1-\lambda) g_0(x)\right)^p && \text{def of } \lambda \\
&= (\alpha+\beta)^p \left(\lambda f_0(x)^p + (1-\lambda) g_0(x)^p\right) && \text{cvxty of } t^p \\
\int |f(x) + g(x)|^p \, d\mu &\leq (\alpha+\beta)^p \int \left(\lambda f_0(x)^p + (1-\lambda) g_0(x)^p\right) d\mu && \text{integrating} \\
\|f+g\|_p^p &\leq (\alpha+\beta)^p \left(\lambda \|f_0\|_p^p + (1-\lambda)\|g_0\|_p^p\right) && \text{linearity} \\
&\leq (\alpha+\beta)^p (\lambda + (1-\lambda)) && \text{by (III.1)} \\
&= (\alpha+\beta)^p \\
\|f+g\|_p &\leq (\alpha+\beta) && p\text{th roots}
\end{aligned}$$

III.2. **The Riesz-Fischer Theorem: $L^p(X, \mathcal{M}, \mu)$ is complete.**

Road map:

Split into the two cases $1 \leq p < \infty, p = \infty$. For each case:

(1) Invoke the Banach characterization lemma.

(2) Define

$$f(x) = \begin{cases} \sum f_n(x) & \text{behaves} \\ 0 & \text{else} \end{cases}$$

Use $g(x) = \sum_k |f_k(x)|$ for $1 \leq p < \infty$, use $N_k = \{x : |f_k(x)| > \|f_k\|_\infty\}$ for $p = \infty$.

(3) Use Minkowski to show $f \in L^p$ and $\sum_{k=1}^n f_k \xrightarrow{L^p} f$.

case i) $1 \leq p < \infty$.

1. By the lemma, it suffices to show that every series which converges absolutely (in $\mathbb{R}$) also converges in $L^p, p \in [1, \infty)$.

2. Let $\sum_{k=1}^\infty \|f_k\|_p < \infty$ for some $\{f_k\}_{k=1}^\infty \subseteq L^p$.
NTS: $\|f - \sum_{k=1}^n f_k\|_p \xrightarrow{n\to\infty} 0$ for some $f \in L^p$, since this is what $\sum_{k=1}^n f_k \xrightarrow{L^p} f$ means.
Define

$$g(x) = \sum_{k=1}^\infty |f_k(x)|$$

so that $g \geq 0$ (g may take the value ∞). Note:

$$\left(\sum_{k=1}^n |f_k|\right)^p \geq 0 \tag{III.2}$$

and since positive exponents preserve order, we also have

$$\left(\sum_{k=1}^n |f_k|\right)^p \leq \left(\sum_{k=1}^{n+1} |f_k|\right)^p. \tag{III.3}$$

Then we have

$$
\begin{aligned}
\|g\|_p &= \left(\int \lim_{n\to\infty} \left| \sum_{k=1}^{n} |f_k| \right|^p d\mu \right)^{1/p} \\
&= \left(\int \lim_{n\to\infty} \left(\sum_{k=1}^{n} |f_k| \right)^p d\mu \right)^{1/p} && \text{by (III.2)} \\
&= \lim_{n\to\infty} \left(\int \left(\sum_{k=1}^{n} |f_k| \right)^p d\mu \right)^{1/p} && \text{by MCT,(III.3)} \\
&= \lim_{n\to\infty} \left\| \sum_{k=1}^{n} |f_k| \right\|_p && \text{def of } \|\cdot\|_p \\
&\le \lim_{n\to\infty} \sum_{k=1}^{n} \|f_k\|_p && \text{Minkowski} \\
&= \sum_{k=1}^{\infty} \|f_k\|_p ,
\end{aligned}
$$

which is finite, by hypothesis. Thus $g \in L^p$, so $|g| \leqslant_{\text{ae}} \infty$. Hence we may define

$$
f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & |g(x)| < \infty \\ 0 & |g(x)| = \infty \end{cases}
$$

so that f is measurable and

$$
|f|^p \le g^p \implies f \in L^p.
$$

Since $\lim_{n\to\infty} |f(x) - \sum_{k=1}^{n} f_k(x)| = 0$ and $|f(x) - \sum_{k=1}^{\infty} f_k(x)|^p \le g^p$ are both true ae, the DCT gives $\|f - \sum_{k=1}^{n} f_k\|_p \xrightarrow{n\to\infty} 0$.

case ii) $p = \infty$.

1. By the lemma, it suffices to show that every series which converges absolutely (in $\mathbb{R}$) also converges in L^∞.

2. Let $\sum_{k=1}^{\infty} \|f_k\|_\infty < \infty$ for some $\{f_k\}_{k=1}^{\infty} \subseteq L^\infty$.
NTS: $\|f - \sum_{k=1}^{n} f_k\|_\infty \xrightarrow{n\to\infty} 0$ for some $f \in L^\infty$.
For each k, define
$$N_k := \{x : |f_k(x)| > \|f_k\|_\infty\},$$
so that $\mu N_k = 0, \forall k \implies \mu\left(\cup_k N_k\right) = 0$. Then if $x \notin \cup_k N_k$,
$$\sum_k |f_k(x)| \leq \sum_k \|f_k\|_\infty \quad \implies \quad \sum_k f_k(x) < \infty,$$
by what we know of $\mathbb{R}$. Now we may define
$$f(x) = \begin{cases} \sum_k f_k(x) & x \notin \cup_k N_k \\ 0 & x \in \cup_k N_k \end{cases}$$
so that f is μ-measurable and bounded, i.e., $f \in L^\infty$.

3. Since $\mu(\cup_k N_k) = 0$,
$$\left\| f - \sum_{k=1}^{n} f_k \right\|_\infty \leq \left\| \sum_{k=n+1}^{\infty} f_k \right\|_\infty \leq \sum_{k=n+1}^{\infty} \|f_k\|_\infty \qquad \text{by Mink}$$
Then taking limits,
$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} f_k \right\|_\infty \leq \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \|f_k\|_\infty = 0.$$
Thus, $\|f - \sum_{k=1}^{n} f_k\|_\infty \xrightarrow{n\to\infty} 0$.

Lemma III.1. (Banach Characterization Lemma).
Suppose that $(X, \|\cdot\|)$ is a normed vector space. Then

$$X \text{ is Banach} \iff \text{every absolutely convergent series in } X \text{ is convergent.}$$

Proof.
($\Rightarrow$) Suppose every Cauchy sequence converges.
Let $\{x_k\}$ be such that $\sum \|x_k\| < \infty$ so $\{x_k\}$ is absolutely convergent.
Then let

$$s_n := \sum_{k=1}^{n} x_k, \quad \text{and} s := \lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} x_k.$$

NTS: $\{s_n\}$ is Cauchy. Wlog, let $n < m$.

$$\begin{aligned}
\|s_n - s_m\| &= \left\| \sum_{k=n+1}^{m} x_k \right\| \\
&\leq \sum_{k=n+1}^{m} \|x_k\| && \triangle\text{-ineq} \\
&\xrightarrow{m\to\infty} \sum_{k=n+1}^{\infty} \|x_k\| \\
&\xrightarrow{n\to\infty} 0
\end{aligned}$$

Hence, $\{s_n\}$ Cauchy implies that $s_n \xrightarrow{n\to\infty} s = \lim_{n\to\infty} s_n \in X$.

($\Leftarrow$) Suppose that every abs. convergent series is convergent.
Let $\{x_n\}$ be a Cauchy sequence.
NTS: $x_n \to x \in X$.
Since $\{x_n\}$ is Cauchy, we can find a subsequence $\{x_{n_k}\}$ which satisfies

$$\|x_{n_k} - x_{n_{k+1}}\| < \tfrac{1}{2^k}.$$

Define

$$v_1 = x_{n_1}, \quad \text{and} \quad v_k = x_{n_{k+1}} - x_{n_k}.$$

Then we have a telescoping sum:

$$\sum_{k=1}^{N} v_k = x_{n_1} + (x_{n_2} - x_{n_1}) + \ldots + (x_{n_N} - x_{n_{N-1}}) = x_{n_N},$$

so

$$\sum_{k=1}^{\infty} \|v_k\| < \sum_{k=1}^{\infty} \tfrac{1}{2^k} = 1$$

shows $\sum_{k=1}^{\infty} \|v_k\|$ converges.
Hence, $\sum_{k=1}^{\infty} v_k$ converges by hypothesis to some $v \in X$. Then

$$\sum_{k=1}^{\infty} v_k = v = \lim_{N\to\infty} \sum_{k=1}^{N} v_k = \lim_{N\to\infty} x_{n_N}$$

shows $x_{n_N} \xrightarrow{N\to\infty} v$. Now

$$\|v - x_n\| \leq \|v - x_{n_k}\| + \|x_{n_k} - x_n\|$$

shows that $x_n \xrightarrow{n\to\infty} v$ also.

□

IV. HILBERT SPACE REVIEW

Most material in this talk is from Reed & Simon.

Definition IV.1. A complex vector space is called an *inner product space* (IPS) when

(i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ iff $x = 0$,

(ii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(iii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$,

(iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

An inner product space is a *Hilbert space* iff it is complete under the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition IV.2. Two vectors $x \neq y$ are *orthogonal* iff $\langle x, y \rangle = 0$.
A collection $\{x_i\}$ is an *orthonormal set* iff

$$\langle x_i, x_i \rangle = 1 \quad \text{and} \quad \langle x_i, x_j \rangle = 0 \quad \forall i \neq j.$$

Proposition IV.3. (Pythagorean Theorem)
Let $\{x_n\}_{n=1}^N$ be an orthogonal set in an IPS. Then

$$\left\| \sum_{n=1}^N x_n \right\|^2 = \sum_{n=1}^N \|x_n\|^2$$

Proof. $\left\| \sum x_n \right\|^2 = \langle \sum x_n, \sum x_n \rangle = \sum_{n,m=1}^N \langle x_n, x_m \rangle$.
Then see that all the terms with $n \neq m$ are 0 because of orthogonality, leaving only $\sum_{n=1}^N \langle x_n, x_n \rangle = \sum_{n=1}^N \|x_n\|^2$. □

Proposition IV.4. (Bessel's Inequality)
If $\{x_\alpha\}_{\alpha \in A}$ is an orthonormal set in an IPS, then for any x,

$$\sum_{\alpha \in A} \left| \langle x, x_\alpha \rangle \right|^2 \leq \|x\|^2.$$

Proof. It suffices to show that $\sum_{\alpha\in F} |\langle x, x_\alpha\rangle|^2 \le \|x\|^2$ for any finite $F \subseteq A$:

$$
\begin{aligned}
0 &\le \left\| x - \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha \right\|^2 \\
&= \left\langle x - \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha, x - \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha \right\rangle \\
&= \|x\|^2 - 2\,\mathrm{Re} \left\langle x, \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha \right\rangle + \left\| \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha \right\|^2 \\
&= \|x\|^2 - 2\,\mathrm{Re} \sum_{\alpha\in F} \langle x, x_\alpha\rangle \overline{\langle x, x_\alpha\rangle} + \left\| \sum_{\alpha\in F} \langle x, x_\alpha\rangle x_\alpha \right\|^2 \\
&= \|x\|^2 - 2 \sum_{\alpha\in F} |\langle x, x_\alpha\rangle|^2 + \sum_{\alpha\in F} |\langle x, x_\alpha\rangle|^2 \qquad \text{(IV.1)} \\
&= \|x\|^2 - \sum_{\alpha\in F} |\langle x, x_\alpha\rangle|^2
\end{aligned}
$$

Where (IV.1) comes by the Pythagorean Thm. □

Note that this theorem indicates $\{\alpha : \langle x, x_\alpha\rangle \neq 0\}$ is countable.

Proposition IV.5. (Schwartz Inequality)
If x and y are vectors in an IPS, then

$$\|x\| \cdot \|y\| \ge |\langle x, y\rangle| .$$

Proof. The case $y = 0$ is trivial, so suppose $y \neq 0$. The vector $\frac{y}{\|y\|}$ by itself forms an orthonormal set, so applying Bessel's inequality to any x gives

$$
\begin{aligned}
\|x\|^2 &\ge \left| \langle x, \tfrac{y}{\|y\|} \rangle \right|^2 \\
&= \frac{|\langle x, y\rangle|^2}{\|y\|^2} \\
\|x\|^2 \|y\|^2 &\ge |\langle x, y\rangle|^2
\end{aligned}
$$

□

Proposition IV.6. $\|x\| = \sqrt{\langle x, x\rangle}$ really is a norm.

Proof. The first two properties of norm are clearly satisfied:

$$\|x\| = 0 \iff x = 0, \quad \|x\| \geq 0,$$

$$\|\alpha x\| = |\alpha| \cdot \|x\|.$$

To see the triangle inequality,

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x+y, x\rangle + \langle x+y, y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle && \text{linearity} \\
&= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 && \tfrac{z+\overline{z}}{2} = \operatorname{Re} z \\
&\leq \|x\|^2 + 2\,|\langle x, y\rangle| + \|y\|^2 && \operatorname{Re} z \leq |z| \\
&\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 && |\langle x, y\rangle| \leq \|x\| \cdot \|y\| \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

□

Proposition IV.7. (Parallelogram Identity)

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Proof. Add the two formulae

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$$
$$\|x-y\|^2 = \|x\|^2 - 2\operatorname{Re}\langle x, y\rangle + \|y\|^2.$$

□

Example.

$$\ell^2 := \left\{ \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

with the inner product

$$\left\langle \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \right\rangle := \sum_{n=1}^{\infty} \overline{x_n}\, y_n.$$

Example.

$$L^2 := \left\{ f : X \to \mathbb{C} : \int_X |f|^2 \, d\mu < \infty \right\}$$

with the inner product

$$\langle f, g \rangle := \int_X \overline{f} g \, d\mu.$$

Example.

$$L^2_{\mathcal{H}} := \left\{ f : X \to \mathcal{H} : \int_X \|f(x)\|^2_{\mathcal{H}} \, d\mu < \infty \right\}$$

with the inner product

$$\langle f, g \rangle := \int_X \langle f(x), g(x) \rangle_{\mathcal{H}} \, d\mu.$$

IV.1. Bases.

Definition IV.8. An *orthonormal basis* of a Hilbert space $\mathcal{H}$ is a maximal orthonormal set S (i.e., no other orthonormal set contains S as a proper subset).

Theorem IV.9. Every Hilbert space has an orthonormal basis.

Proof. Let $\mathcal{C}$ be the collection of orthonormal subsets of $\mathcal{H}$. Order $\mathcal{C}$ by inclusion:

$$S_1 \prec S_2 \quad \Longleftrightarrow \quad S_1 \subseteq S_2.$$

Then $(\mathcal{C}, \prec)$ is a poset.
It is also nonempty since $\{x/\|x\|\}$ is an orthonormal set, $\forall x \in \mathcal{H}$.
Now let $\{S_\alpha\}_{\alpha \in A}$ be any linearly ordered subset of $\mathcal{C}$.
Then $\cup_{\alpha \in A} S_\alpha$ is an orthonormal set which contains each S_α and is thus an upper bound for $\{S_\alpha\}_{\alpha \in A}$.
Since every linearly ordered subset of $\mathcal{C}$ has an upper bound, apply Zorn's Lemma and conclude that $\mathcal{C}$ has a maximal element.
This maximal element is an orthonormal set not properly contained in any other orthonormal set. □

Theorem IV.10. (Orthogonal Decomposition and Parseval's Relation)
Let $S = \{x_\alpha\}_{\alpha \in A}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then $\forall y \in \mathcal{H}$:

$$y = \sum_{\alpha \in A} \langle x_\alpha, y \rangle x_\alpha, \quad \text{and} \quad \|y\|^2 = \sum_{\alpha \in A} |\langle x_\alpha, y \rangle|^2.$$

Proof. Proving Bessel's inequality, we saw that

$$\sum_{\alpha \in A} |\langle x_\alpha, y \rangle|^2 \leq \|y\|^2,$$

and that there are at most countably many nonzero summands.
Collect these α's for which $\langle x_\alpha, y \rangle \neq 0$ to obtain a sequence $\{\alpha_j\}_{j=1}^\infty$.
As a positive-term series, $\sum_{j=1}^N |\langle x_{\alpha_j}, y \rangle|^2$ is monotone increasing.
It is also bounded above by $\|y\|^2$.
Thus, it converges to a finite limit as $N \to \infty$. Define

$$y_n := \sum_{j=1}^n \langle x_{\alpha_j}, y \rangle x_{\alpha_j}.$$

We want to show $\lim y_n = y$. For $n > m$,

$$\|y_n - y_m\|^2 = \left\| \sum_{j=m+1}^n \langle x_{\alpha_j}, y \rangle x_{\alpha_j} \right\|^2 = \sum_{j=m+1}^n |\langle x_{\alpha_j}, y \rangle|^2,$$

by the Pythagorean Thm. Letting $n, m \to \infty$ shows $\{y_n\}$ is Cauchy.
Since $\mathcal{H}$ is Hilbert, it is complete and $\{y_n\}$ must converge to some $y' \in \mathcal{H}$.
Let x_α be any element of S. If $\exists \ell \; \alpha = \alpha_\ell$, then by the continuity of norms:

$$\langle y - y', x_{\alpha_\ell} \rangle = \lim_{n \to \infty} \left\langle y - \sum_{j=1}^n \langle x_{\alpha_j}, y \rangle x_{\alpha_j}, x_{\alpha_\ell} \right\rangle = \langle y, x_{\alpha_\ell} \rangle - \langle y, x_{\alpha_\ell} \rangle = 0$$

and if not,

$$\langle y - y', x_\alpha \rangle = \lim_{n \to \infty} \left\langle y - \sum_{j=1}^n \langle x_{\alpha_j}, y \rangle x_{\alpha_j}, x_\alpha \right\rangle = 0$$

because

$$
\begin{aligned}
\langle y - y', x_\alpha \rangle &= \lim_{n\to\infty} \left\langle y - \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle x_{\alpha_j}, x_\alpha \right\rangle \\
&= \langle y, x_\alpha\rangle - \lim_{n\to\infty} \sum_{j=1}^{n} \big\langle \langle x_{\alpha_j}, y\rangle x_{\alpha_j}, x_\alpha \big\rangle \\
&= 0 - \sum_{j=1}^{\infty} \big\langle \langle x_{\alpha_j}, y\rangle x_{\alpha_j}, x_\alpha \big\rangle && \langle y, x_\alpha\rangle = 0 \text{ for } \alpha \neq \alpha_\ell \\
&= \sum_{j=1}^{\infty} \langle y, x_{\alpha_j}\rangle \left\langle x_{\alpha_j}, x_\alpha \right\rangle \\
&= \sum_{j=1}^{\infty} \langle y, x_{\alpha_j}\rangle \cdot 0 && \langle x_{\alpha_j}, x_\alpha\rangle = 0 \text{ for } \alpha \neq \alpha_\ell \\
&= 0
\end{aligned}
$$

So $y - y'$ is orthogonal to every x_α in S. Since S is a orthonormal basis, this means we must have $y - y' = 0$. Thus

$$
y = \lim_{n\to\infty} \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle x_{\alpha_j},
$$

and we have shown the first part. Finally,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \left\| y - \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle x_{\alpha_j} \right\|^2 \\
&= \lim_{n\to\infty} \left(\|y\|^2 - 2\,\mathrm{Re} \left\langle y, \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle x_{\alpha_j} \right\rangle + \left\| \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle x_{\alpha_j} \right\|^2 \right) \\
&= \lim_{n\to\infty} \left(\|y\|^2 - 2\,\mathrm{Re} \sum_{j=1}^{n} \langle x_{\alpha_j}, y\rangle \left\langle y, x_{\alpha_j} \right\rangle + \sum_{j=1}^{n} \left\| \langle x_{\alpha_j}, y\rangle x_{\alpha_j} \right\|^2 \right) \\
&= \lim_{n\to\infty} \left(\|y\|^2 - 2 \sum_{j=1}^{n} \left| \langle x_{\alpha_j}, y\rangle \right|^2 + \sum_{j=1}^{n} \left| \langle x_{\alpha_j}, y\rangle \right|^2 \left\| x_{\alpha_j} \right\|^2 \right)
\end{aligned}
$$

$$= \lim_{n\to\infty}\left(\|y\|^2 - \sum_{j=1}^{n} |\langle x_{\alpha_j}, y\rangle|^2\right)$$
$$= \|y\|^2 - \sum_{\alpha\in A} |\langle x_\alpha, y\rangle|^2$$

gives Parseval's Relation:

$$\|y\|^2 = \sum_{\alpha\in A} |\langle x_\alpha, y\rangle|^2.$$

□

Definition IV.11. The coefficients $\langle x_\alpha, y\rangle$ are the *Fourier coefficients* of y with respect to the basis $\{x_\alpha\}$.

IV.2. **The Riesz Representation Theorem *Again*.**

Definition IV.12. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then $\mathcal{M}$ is a Hilbert space under the inner product it inherits as a subspace of $\mathcal{H}$. Define the *orthogonal complement* of $\mathcal{M}$ to be

$$\mathcal{M} := \{x \in \mathcal{H} \vdots \langle x, y\rangle = 0 \; \forall y \in \mathcal{M}\}.$$

Theorem IV.13. (Projection Theorem)
If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. That is, $\forall x \in \mathcal{H}$, x can be uniquely expressed as $x = y + z$, where $y \in \mathcal{M}, z \in \mathcal{M}^\perp$. Moreover, y, z are the unique elements of $\mathcal{M}$ and $\mathcal{M}^\perp$ whose distance to x is minimal.

If $y \in \mathcal{H}$, then $\varphi_y(x) = \langle x, y\rangle$ defines a functional on $\mathcal{H}$.
By the linearity of inner prod, it is a linear functional.
By the Schwartz inequality[8],

$$\|\varphi_y\| = \sup_{\|x\|\le 1} \|\varphi_y(x)\| = \sup_{\|x\|\le 1} |\langle x, y\rangle| \le \sup_{\|x\|\le 1} \|x\|\cdot\|y\| \le \|y\|$$

Shows that this functional is bounded/ continuous.

[8]Recall, the Schwartz ineq is just the Hölder ineq when $p = 2$.

Theorem IV.14. (Riesz Representation Theorem for Hilbert Spaces) If $\varphi \in \mathcal{H}^*$, then $\exists! y \in \mathcal{H}$ such that $\varphi(x) = \langle x, y \rangle \; \forall x \in \mathcal{H}$. Also, $\|\varphi\| = \|y\|$.

Proof. If φ is the zero functional, then $y = 0$ and we're done.
Otherwise, consider the nullspace

$$\mathcal{M} := \{x \in \mathcal{H} \vdots \varphi(x) = 0\}.$$

$\mathcal{M}$ is a proper closed subspace of $\mathcal{H}$ and $\mathcal{M}^\perp \neq \{0\}$ by the Projection Thm. Thus we can find $z \in \mathcal{M}^\perp$ with $\|z\| = 1$ and define

$$u := \varphi(x)z - \varphi(z)x.$$

Then

$$\varphi(u) = \varphi\Big(\varphi(x)z - \varphi(z)x\Big) = \varphi(x)\varphi(z) - \varphi(z)\varphi(x) = 0$$

shows that $u \in \mathcal{M}$ and hence that $u \perp z$. Thus,

$$\begin{aligned} 0 = \langle z, u \rangle &= \langle z, \varphi(x)z - \varphi(z)x \rangle \\ &= \langle z, \varphi(x)z \rangle - \langle z, \varphi(z)x \rangle && \text{linearity} \\ &= \varphi(x)\|z\|^2 - \varphi(z)\langle z, x \rangle && \langle z, z \rangle = \|z\|^2 \\ &= \varphi(x) - \langle x, \varphi(z)z \rangle && \|z\| = 1 \end{aligned}$$

Thus, $\varphi(x) = \langle x, y \rangle$ where $y = \varphi(z)z$.
As for uniqueness, if $\langle x, y \rangle = \langle x, y' \rangle$ for all x, take $x = y - y'$ and get

$$\|y - y'\|^2 = \langle y - y', y - y' \rangle = \langle y - y', y \rangle - \langle y - y', y' \rangle = 0 \quad \implies \quad y = y'.$$

□

This shows that $y \mapsto \varphi_y$ is a conjugate linear isometry of $\mathcal{H}$ onto $\mathcal{H}^*$.

Definition IV.15. Isomorphisms of Hilbert spaces are those transformations $U : \mathcal{H}_1 \to \mathcal{H}_2$ which preserve the inner product:

$$\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1} \; \forall x, y \in \mathcal{H}_1.$$

Such operators are called *unitary*.
For $U : \mathcal{H} \to \mathcal{H}$, unitary operators are also characterized by $U^* = U^{-1}$, where T^* is the Hilbert space adjoint of $T \in L(\mathcal{H})$ and is defined by

$$\langle T^*x, y \rangle = \langle x, Ty \rangle.$$

V. A Practical Guide to Integral Problems

This talk covers the relation between Riemann and Lebesgue integration, when you can differentiate under an integral, and other practical applications of Lebesgue theory to standard integral problems.

V.1. Some related theorems.

Theorem V.1. Let $f : [a, \infty) \to \mathbb{R}$ be locally R-integrable. Then

$$f \in L^1[a,\infty) \iff {}^{\mathrm{R}}\!\!\int_a^\infty |f|\, dx < \infty \text{ and } {}^{\mathrm{R}}\!\!\int_a^\infty f\, dx = {}^{\mathrm{L}}\!\!\int_{[a,\infty)} f\, d\mu.$$

Proof. ($\Rightarrow$) $f \in L^1 \implies f^+, f^- \in L^1$.
Define $f_n = f^+ \chi_{[a,a+n)}$ so that $f_n \leq f_{n+1}$, and $f_n \to f^+$ and $f_n \in L^1$.
Now for $A := [a, \infty)$,

$$\begin{aligned}
{}^{\mathrm{R}}\!\!\int_a^\infty f^+ dx &= \lim_{n\to\infty} {}^{\mathrm{R}}\!\!\int_a^{a+n} f^+ dx && \text{def of improper int} \\
&= \lim_{n\to\infty} {}^{\mathrm{L}}\!\!\int_{[a,a+n)} f^+ dx && R = L \text{ on bounded} \\
&= \lim_{n\to\infty} {}^{\mathrm{L}}\!\!\int_A f_n\, d\mu && \text{def } f_n \\
&= {}^{\mathrm{L}}\!\!\int_A \lim_{n\to\infty} f_n\, d\mu && \text{MCT} \\
&= {}^{\mathrm{L}}\!\!\int_A f^+\, d\mu
\end{aligned}$$

Similarly, ${}^{\mathrm{R}}\!\int_a^\infty f^- dx = {}^{\mathrm{L}}\!\int_A f^-\, d\mu$, so

$${}^{\mathrm{L}}\!\!\int_A f\, d\mu = {}^{\mathrm{L}}\!\!\int_A \left(f^+ - f^-\right) d\mu = {}^{\mathrm{R}}\!\!\int_a^\infty \left(f^+ - f^-\right) dx = {}^{\mathrm{R}}\!\!\int_a^\infty f\, dx.$$

($\Leftarrow$) Define $f_n(x) = |f|\chi_{[a,a+n]}$ so that $f_n \nearrow |f|$ and f_n is R-integrable. (support is compact)

Then ${}^{\mathrm{R}}\!\int_a^{[a+n]} f_n dx$ exists and ${}^{\mathrm{R}}\!\int_a^{[a,a+n]} f_n dx = {}^{\mathrm{L}}\!\int_{[a,a+n]} f_n d\mu$, so

$$\begin{aligned}
{}^{\mathrm{L}}\!\!\int_A |f|\, d\mu &= \lim_{n\to\infty} {}^{\mathrm{L}}\!\!\int_A f_n\, d\mu && \text{MCT} \\
&= \lim_{n\to\infty} {}^{\mathrm{R}}\!\!\int_a^{a+n} f_n dx && R = L \text{ on bounded} \\
&= \lim_{n\to\infty} {}^{\mathrm{R}}\!\!\int_a^{a+n} |f| dx && \text{def of } f_n \\
&= {}^{\mathrm{R}}\!\!\int_a^{\infty} |f| dx && \text{def of impr int} \\
&< \infty && \text{hypothesis}
\end{aligned}$$

shows that $f \in L^1$. □

Theorem V.2. Define $F(t) = \int_X f(x,t)\, d\mu(x)$ for $f : X \times [a,b] \to \mathbb{C}$.

(1) What is sufficient for F to be continuous? $\lim_{t\to t_0} F(t) = F(t_0)$, $\forall t_0$.

(2) What is sufficient for F to be differential? $F'(t) = \displaystyle\int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x)$.

Royden:

(1) (i) $f_t(x) = f(x,t)$ is a measurable function of x for each fixed t.

(ii) $\forall t, |f(x,t)| \le g(x) \in L^1(X)$.

(iii) $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for each x (i.e., $f(x,t)$ is continuous in t for each x).

The proof follows by applying DCT to $f(x,t_n)$, where $t_n \to t_0$.

(2) (i) $\frac{\partial f}{\partial t}$ exists on $X \times [a,b]$,

(ii) $\frac{\partial f}{\partial t}$ is bounded on $X \times [a,b]$,

(iii) f is bounded on $X \times [a,b]$,

(iv) For each fixed t, f is a measurable function of x.

(3) Alternatively:

(i) $\frac{\partial f}{\partial t}$ exists on $X \times [a, b]$,

(ii) $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x) \in L^1(X)$ on $X \times [a, b]$.

advantages: $f, \frac{\partial f}{\partial t}$ need not be bounded
disadvantages: need $f \in L^1$.

Proof. (of the second version).
Pick any sequence $\{t_n\} \subseteq [a, b]$ with $t_n \to t_0$. Then define

$$h_n(x) := \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}.$$

Then $\frac{\partial f}{\partial t}(x,t_0) = \lim_{n\to\infty} h_n(x)$, so $\frac{\partial f}{\partial t}(x,t_0)$ is measurable as a limit of measurable functions. It follows that $\frac{\partial f}{\partial t}(x,t)$ is measurable. By the mean value theorem, there is a t between t_n and t_0 for which

$$f(x,t_n) - f(x,t_0) = (t_n - t_0)\tfrac{\partial f}{\partial t}(x,t).$$

Then

$$|h_n(x)| \leq \sup_{t\in[a,b]} \left|\tfrac{\partial f}{\partial t}(x,t)\right| \leq g(x),$$

since taking the supremum can only make it larger.
Invoke the dominated convergence theorem again and get

$$F'(t_0) = \lim \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim \int h_n(x)\, d\mu(x) = \int \tfrac{\partial f}{\partial t}(x,t) d\mu(x).$$

Finally, exploit the compactness of $[a, b]$ and [Rudin 4.2]:

$$\lim_{x\to t} g(x) = g(t) \quad \iff \quad \lim_{n\to\infty} g(x_n) = g(t), \quad \forall \{x_n\} \subseteq X, x_n \to t.$$

□

V.2. **Solutions to the Nasty Integrals.**

1. $f : X \to [0, \infty]$ is measurable and $\int_X f\, d\mu = c$ where $0 < c < \infty$. Let $\alpha \in \mathbb{R}$ be a constant. Show that

$$\lim_{n\to\infty} \int_X n \log \left[1 + \left(\frac{f(x)}{n}\right)^\alpha\right] d\mu = \begin{cases} \infty & 0 < \alpha < 1 \\ c & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$$

Proof.

case i) $\alpha = 1$.

By basic calculus, $\left(1 + \frac{f(x)}{n}\right)^n$ increases to $e^{f(x)}$ for each x, so

$$g_n(x) = \log \left(1 + \tfrac{f(x)}{n}\right)^n \xrightarrow{n\to\infty} f(x) \in L^1 \text{ (increasing)}.$$

Then $\lim_{n\to\infty} \int_X g_n(x) = \int_X f(x) = c$ by MCT.

case ii) $\alpha > 1$.

Note that $f(x) \geq 0$, $\int_X f\, d\mu = c \leq \infty$ show f is finite μ-ae, i.e.:

$$\exists M < \infty \text{ and } \exists E \in \mathcal{M} \text{ s.t. } \mu E = 0 \text{ and } \left| f\big|_{\tilde{E}}(x) \right| \leq M,$$

(so E is where f is bounded). Since we can always find N such that $n \geq N \implies \frac{M}{n} < 1$, $\frac{f(x)}{n} \leq \frac{M}{n}$, μ-au. We're concerned with $n \to \infty$, so this means $\frac{f(x)}{n} < 1$ for our purposes. Hence, $\alpha > 1 \implies \alpha - 1 > 0$ implies

$$0 \leq \left(\tfrac{f(x)}{n}\right)^{\alpha-1} < 1. \tag{V.1}$$

We need

$$n \log \left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) \leq \alpha f(x),$$

so define

$$G(t) = n \log \left(1 + \left(\tfrac{t}{n}\right)^\alpha\right) - \alpha t.$$

Then $G(0) = 0$. Also,

$$\begin{aligned} G'(t) &= \left(\frac{nt^{\alpha-1}}{n^\alpha + t^\alpha} - 1\right)\alpha && \text{diff} \\ &\le \left(\frac{nt^{\alpha-1}}{n^\alpha} - 1\right)\alpha && \text{drop the } t^\alpha \\ &= \left(\left(\frac{t}{n}\right)^{\alpha-1} - 1\right)\alpha && \text{simp} \\ &< 0 && t = f(x), \left(\frac{f(x)}{n}\right)^{\alpha-1} < 1, \end{aligned}$$

so G is decreasing and $G(t) < 0$ for $t > 0$, i.e.,

$$n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) \le \alpha f(x).$$

Set

$$g_n(x) = n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right).$$

Since this is bounded by αf and $f \in L^1$ by hypothesis, DCT gives

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X \lim_{n\to\infty} g_n \, d\mu.$$

Now split the leading n and match the denominator:

$$g_n(x) = n^{1-\alpha} n^\alpha \log\left(1 + \left(\frac{f}{n}\right)^\alpha\right) = n^{1-\alpha} \log\left(1 + \left(\frac{f^\alpha}{n^\alpha}\right)\right)^{n^\alpha},$$

so that

$$\lim_{n\to\infty} \left(1 + \frac{f^\alpha}{n^\alpha}\right)^{n^\alpha} = e^f \leqslant_{\text{ae}} e^M$$

shows

$$\lim_{n\to\infty} \frac{\log\left(1 + \frac{f^\alpha}{n^\alpha}\right)^{n^\alpha}}{n^{\alpha-1}} = 0.$$

case iii) $\alpha < 1$.

$$f > 0 \implies \log\left(1 + \frac{f^\alpha}{n^\alpha}\right) > 0,$$

and if we define $A := \{f > 0\}$, then $\mu A > 0$ because $\int_X f \, d\mu > 0$. Thus $\int_A f \, d\mu = \int_X f \, d\mu$, and

$$\log\left(1 + \frac{f^\alpha}{n^\alpha}\right) > 0 \quad \text{on } A.$$

But then

$$\frac{\log\left(1+\frac{f^\alpha}{n^\alpha}\right)^{n^\alpha}}{n^{\alpha-1}} \xrightarrow{n\to\infty} \infty$$

because $\alpha < 1 \implies \alpha - 1 < 0$.

Thus, $\underline{\lim}\, g_n = \infty$, so by Fatou's Lemma,

$$\underline{\lim} \int g_n \, d\mu \geq \int \underline{\lim}\, g_n = \infty \quad \implies \quad \lim_{n\to\infty} \int g_n \, d\mu = \infty.$$

□

2. Define $F(t) = \displaystyle\int_0^\infty \frac{e^{-xt}}{1+x^2}dx$, for $t > 0$.

a) Show that F is well-defined as an improper Riemann integral and as a Lebesgue integral.

Riemann: $\frac{e^{-xt}}{1+x^2}$ is continuous $\forall t$, so it is R-integrable on any bounded interval (a, b). So only remains to show the convergence of

$$\lim_{a\to\infty} \int_0^a \tfrac{e^{-xt}}{1+x^2}dx.$$

Since the integrand is nonnegative,[9]

$$b \geq a \quad \Longrightarrow \quad \int_0^a \tfrac{e^{-xt}}{1+x^2}dx \leq \int_0^b \tfrac{e^{-xt}}{1+x^2}dx.$$

Thus, it suffices to consider a dominating function $g(x)$:

$$\tfrac{e^{-xt}}{1+x^2} \leq \tfrac{1}{1+x^2} \leq \tfrac{1}{x^2} \quad \forall t > 0.$$

Since

$$\int_1^a \tfrac{1}{x^2}dx = \left[-\tfrac{1}{x}\right]_1^a = -\tfrac{1}{a} + 1 \xrightarrow{a\to\infty} 1$$

and the integrand is bounded by 1, $\forall t > 0$, we have

$$\int_0^a \tfrac{e^{-xt}}{1+x^2}dx \leq 2 \ \forall a,$$

and thus it converges as $a \to \infty$.

Lebesgue: $\int_{\mathbb{R}^+} \left|\frac{e^{-xt}}{1+x^2}\right| d\mu$ exists because we can bound the integrand as above.

b) Show $F''(t)$ exists on $(0, \infty)$.

We have $\varphi(t) = \frac{e^{-xt}}{1+x^2} \in L^1$, so just need $\frac{\partial\varphi}{\partial t}(t) = -\frac{xe^{-xt}}{1+x^2} \in L^1$ in order to use the theorem and get

$$F'(t) = -\int_0^\infty \frac{xe^{-xt}}{1+x^2}dx.$$

[9] $f(x) \geq 0 \implies g(x) = \int_0^x f(t)\,dt$ is increasing.

Fix $t > 0$ and pick $\varepsilon > 0$ such that $t - \varepsilon > 0$.
Observe: $x \le e^{\varepsilon x}$ for large enough x. Thus we pick M large enough that $x \ge M \implies x \le e^{\varepsilon x}$, and split the integral:

$$\int_0^\infty \frac{-xe^{-xt}}{1+x^2}\,dx = \int_0^M \frac{-xe^{-xt}}{1+x^2}\,dx + \int_M^\infty \frac{-xe^{-xt}}{1+x^2}\,dx.$$

Then we have

$$\int_0^M \left|\frac{xe^{-xt}}{1+x^2}\right|\,dx \le \int_0^M \left|xe^{-xt}\right|\,dx,$$

which as a continuous function over a compact space is clearly finite, and hence the integrand is in $L^1(0,M)$. Also,

$$\begin{aligned}
\int_M^\infty \left|\frac{-xe^{-xt}}{1+x^2}\right|\,dx &\le \int_M^\infty \left|\frac{e^{\varepsilon x}e^{-xt}}{1+x^2}\right|\,dx && \text{by choice of } M \\
&= \int_M^\infty \left|\frac{e^{(\varepsilon-t)x}}{1+x^2}\right|\,dx \\
&= \int_0^\infty \frac{e^{(\varepsilon-t)x}}{1+x^2}\,dx && \text{positive integrand}
\end{aligned}$$

So $\varepsilon - t < 0 \implies \frac{e^{(\varepsilon-t)x}}{1+x^2} \in L^1$ by (a).
Thus $\frac{-xe^{-tx}}{1+x^2} \in L^1$. Now if $\frac{\partial}{\partial t}\left(\frac{-xe^{-tx}}{1+x^2}\right) = \frac{x^2e^{-tx}}{1+x^2} \in L^1$, we'll have

$$F''(t) = \int_0^\infty \frac{x^2e^{-tx}}{1+x^2}\,dx.$$

To show this, find N such that $x \ge N \implies x^2 \le e^{\varepsilon x}$, and proceed as before.

c) (Extra credit) Show $F(t)$ satisfies $F''(t)+F(t) = \frac{1}{t}$. Compute $F(t)$.

We have $F''(t) = \int_0^\infty \frac{x^2 e^{-tx}}{1+x^2}\,dx$ from (b), so

$$\begin{aligned} F''(t) + F(t) &= \int_0^\infty \frac{x^2 e^{-tx} + e^{-tx}}{1+x^2}\,dx \\ &= \int_0^\infty \frac{(1+x^2)e^{-tx}}{1+x^2}\,dx \\ &= \int_0^\infty e^{-tx}\,dx \\ &= \left[-\tfrac{1}{t}e^{-tx}\right]_0^\infty = 0 - (-\tfrac{1}{t}) = \tfrac{1}{t}. \end{aligned}$$

Now solve the differential equation $F''(t) + F(t) = \frac{1}{t}$.

3. Let I be an open interval of $\mathbb{R}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto e^{xt} f(x)$ is integrable for each fixed $t \in I$. Define $F : I \to \mathbb{R}$ by

$$F(t) = \int_{\mathbb{R}} e^{xt} f(x)\,dx.$$

Show that F is differentiable with derivative $F'(t) = \int_{\mathbb{R}} xe^{xt} f(x)\,dx$ at each $t \in I$.

Note: $xe^{tx}f(x)$ may not be in L^1! We would like to compute $F'(t)$ by

$$F'(t_0) = \lim_{n\to\infty} \int \left(\frac{e^{t_n x} - e^{t_0 x}}{t_n - t_0}\right) f(x)\,dx, \tag{V.2}$$

where $\{t_n\}$ is a sequence in I with $t_n \to t_0$.
To use DCT, we need to find $g \in L^1$ such that

$$\left|\frac{e^{t_n x} - e^{t_0 x}}{t_n - t_0}\right| \le g(x) \quad \forall n.$$

Choose $t' \in I$ such that $t_n < t'$, $\forall n$. This is possible, since otherwise there would be a subsequence of $\{t_n\}$ converging to $\sup\{t \in I\}$.
↯ I is open and $t_n \to t_0 \in I$.
By MVT,

$$\left|e^{t_n x} - e^{t_0 x}\right| \le |xe^{sx}| \cdot |t_n - t_0| \text{ for some } s \in [t_n, t_0].$$

Since $s < t'$,

$$\left|\frac{e^{t_n x} - e^{t_0 x}}{t_n - t_0}\right| \leq \left|xe^{t'x}\right| \implies \left|\frac{e^{t_n x} - e^{t_0 x}}{t_n - t_0} f(x)\right| \leq \left|xe^{t'x} f(x)\right|, \tag{V.3}$$

and we have a bound which no longer depends on n.

To see that $g(x) = \left|xe^{t'x} f(x)\right|$ is integrable, split the integral: pick $\varepsilon > 0$ such that $t' + \varepsilon \in I$ and choose M be such that

$$x \geq M \quad \implies \quad x \leq e^{\varepsilon x}.$$

Now

$$\int_0^M \left|xe^{t'x} f(x)\right| \, dx \leq M \int_0^M \left|e^{t'x} f(x)\right| \, dx < \infty \qquad t' \in I, \text{ and}$$
$$\int_M^\infty \left|xe^{t'x} f(x)\right| \, dx \leq \int_M^\infty \left|e^{(t'+\varepsilon)x} f(x)\right| \, dx < \infty \qquad t' + \varepsilon \in I.$$

Thus we have $\int_0^\infty \left|xe^{t'x} f(x)\right| \, dx < \infty$. For $\int_{-\infty}^0 g(x)\, dx$, pick $\varepsilon > 0$ such that $t' - \varepsilon \in I$ and let M be such that

$$x \geq M \quad \implies \quad x \leq e^{\varepsilon x},$$

and proceed as for $\int_0^\infty g(x)\, dx$.
Together, this gives $g \in L^1$. By (V.3), we can use the DCT in (V.2) to obtain the result.

4. [2003] Let f be a bounded measurable function on $[0, \infty)$. Show that

$$F(t) = \int_0^\infty \frac{f(x)e^{-xt}}{\sqrt{x}} \, dx, \quad t > 0$$

is continuously differentiable on $(0, \infty)$.

Let us denote the integrand by $\varphi(x, t) := f(x)e^{-xt}x^{-1/2}$.
We would like to find $F'(t)$ by choosing any sequence $\{t_n\}$ such that

$t_n \to t_0$ and computing

$$F'(t_0) = \lim_{n\to\infty} \int_0^\infty \left(\frac{f(x)e^{-xt_n}}{t_n\sqrt{x}} - \frac{f(x)e^{-xt_0}}{t_0\sqrt{x}} \right) dx$$
$$= \lim_{n\to\infty} \int_0^\infty f(x)\frac{e^{-xt_n} - e^{-xt_0}}{t_n - t_0} x^{-1/2}\, dx$$

Since f is bounded, $|f(x)| \le M$. Then

$$\left| f(x)\frac{e^{-xt_n} - e^{-xt_0}}{t_n - t_0} x^{-1/2} \right| \le M \left| \frac{e^{-xt_n} - e^{-xt_0}}{t_n - t_0} x^{-1/2} \right|.$$

Since t ranges over $(0, \infty)$ and $t_n \to t_0 < \infty$, we can certainly pick a strict lower bound τ of $\{t_n\}$, i.e. a number τ such that $\inf\{t_n\} > \tau$. By MVT,

$$\left| e^{-t_n x} - e^{-t_0 x} \right| \le \left| x e^{-sx} \right| \cdot |t_n - t_0| \text{ for some } s \in [t_n, t_0].$$

Since $s > \tau$,

$$\left| \frac{e^{-t_n x} - e^{-t_0 x}}{t_n - t_0} x^{-1/2} \right| \le \left| x e^{-sx} x^{-1/2} \right| \le e^{-\tau x} x^{1/2} \in L^1.$$

To verify the integrability of the dominating function, note that

$$u = x^{1/2} \qquad\qquad dv = e^{-x} dx$$
$$2du = x^{-1/2} dx \qquad\qquad v = -e^{-x}$$

gives

$$\int e^{-x} x^{1/2} = \left[-e^{-x} x^{1/2} \right]_0^\infty + 2\int -e^{-x} x^{-1/2} dx$$
$$= (0 - 0) + 4\int_0^\infty e^{-u^2} du \qquad \text{put } u = \sqrt{x}$$
$$= 4 \cdot \tfrac{\sqrt{\pi}}{2}$$
$$= 2\sqrt{\pi}.$$

Further, $\int_0^\infty e^{-x}x^{-1/2}dx = 2\sqrt{\pi} \implies \int_0^\infty e^{-\tau x}x^{-1/2}dx = 2\sqrt{\frac{\pi}{\tau}}$. Thus

$$\begin{aligned}
F'(t_0) &= \lim_{n\to\infty}\int_0^\infty f(x)\frac{e^{-xt_n}-e^{-xt_0}}{t_n-t_0}x^{-1/2}\,dx \\
&= \int_0^\infty \lim_{n\to\infty} f(x)\frac{e^{-xt_n}-e^{-xt_0}}{t_n-t_0}x^{-1/2}\,dx \qquad \text{by DCT} \\
&= \int_0^\infty f(x)\lim_{t\to t_0}\left(\frac{e^{-xt}-e^{-xt_0}}{t-t_0}\right)x^{-1/2}\,dx \\
&= \int_0^\infty f(x)\left(\tfrac{\partial}{\partial t}e^{-xt}\right)x^{-1/2}\,dx \\
&= \int_0^\infty f(x)\left(-xe^{-xt}\right)x^{-1/2}\,dx \\
&= -\int_0^\infty f(x)e^{-xt}x^{1/2}\,dx.
\end{aligned}$$

Let us denote this function

$$G(t) = -\int_0^\infty f(x)e^{-xt}x^{1/2}\,dx.$$

Since we are required to show that F is continuously differential, we must show that G is continuous.

Notice that another u-substitution with $u = (tx)^{1/2}$, $\frac{2}{t}u\,du = dx$ allows us to rewrite

$$G(t) = -\tfrac{2}{t}\int_0^\infty f(x)xe^{-x^2}dx.$$

Thus, all we need to do is show the integral to be finite. Since f is bounded by M, it suffices to show

$$\int_0^\infty xe^{-x^2}\,dx < \infty.$$

Next, putting $u = x^2$, $\frac{1}{2}du = x\,dx$,

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= \tfrac{1}{2}\int_0^\infty e^{-u}\,du \\ &= \tfrac{1}{2}\left[-e^{-u}\right]_0^\infty \\ &= \tfrac{1}{2}\left(0-(-1)\right) \\ &= \tfrac{1}{2}\end{aligned}$$

which is as finite as it gets.

5. [2000] Show $F(t) = \displaystyle\int_{-\infty}^{\infty} \frac{\sin(x^2 t)}{1+x^2}\,dx$ is continuous on $\mathbb{R}$.

First, note that $|\sin x| \le 1$ gives

$$\left|\frac{\sin(x^2t)}{1+x^2}\right| \le \frac{1}{1+x^2} \in L^1,$$

where the final inclusion is clear from

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = [\arctan x]_{-\infty}^{\infty} = \tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right) = \pi.$$

Pick any $\{t_n\} \in \mathbb{R}$ with $t_n \to t_0$. Then

$$\begin{aligned}\lim_{n\to\infty} F(t_n) &= \lim_{n\to\infty}\int_{-\infty}^{\infty} \frac{\sin(x^2t_n)}{1+x^2}\,dx && \text{def of } F \\ &= \int_{-\infty}^{\infty} \lim_{n\to\infty}\frac{\sin(x^2t_n)}{1+x^2}\,dx && \text{DCT} \\ &= \int_{-\infty}^{\infty} \frac{\sin(x^2\lim_{n\to\infty}t_n)}{1+x^2}\,dx && \text{contin of } \sin x\text{, mult by } x^2 \\ &= \int_{-\infty}^{\infty} \frac{\sin(x^2t_0)}{1+x^2}\,dx && t_n \to t_0 \\ &= F(t_0).\end{aligned}$$

Since this is true for all sequences $t_n \to t_0$, we have $\lim_{t\to t_0} F(t) = F(t_0)$.

6. [1998] $f \in C[0,1]$ is such that $\int_0^1 x^n f(x)\,dx = 0$ for $n = 0,1,2,\ldots$. Show that $f \equiv 0$.

By the Stone-Weierstraß Theorem, there is a sequence of polynomials $\{P_k(x)\}$ such that $P_k(x) \xrightarrow{\text{unif}} f(x)$. Then

$$\int P_k(x)f(x)\,dx \xrightarrow{k\to\infty} \int [f(x)]^2\,dx \qquad \text{by uniformity.}$$

But since any polynomial may be written

$$P(x) = \sum_{i=0}^{m} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m,$$

the linearity of the integral and the hypothesis $\int_0^1 x^n f(x)\,dx = 0$ give

$$\begin{aligned}\int_0^1 P(x)f(x)\,dx &= a_0\int_0^1 dx + a_1\int_0^1 x\,dx + \cdots + a_m\int_0^1 x^m\,dx \\ &= a_0\cdot 0 + a_1\cdot 0 + a_2\cdot 0 + \cdots + a_m\cdot 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\text{So} \quad \int P_k(x)f(x)dx &= 0 \quad \forall k \\ \int f^2 dx &= 0 \\ f^2 &\equiv 0 \qquad\qquad f^2 \geq 0 \\ f &\equiv 0\end{aligned}$$

7. Compute the limits

a) $\lim\limits_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\,dx$

Note that

$$\left|\sin\left(\tfrac{x}{n}\right)\right| \leq 1 \quad \text{and} \quad \left(1+\tfrac{x}{n}\right)^{-n} \xrightarrow{n\to\infty} e^{-x}.$$

Then $\left|\left(1+\frac{x}{n}\right)^{-n}\right| \leq \left(1+\frac{x}{2}\right)^{-2} \quad \forall n \geq 2$, so the DCT gives

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx &= \int_0^\infty \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx \\
&= \int_0^\infty e^{-x} \sin(0)\, dx \\
&= 0.
\end{aligned}$$

b) $\lim\limits_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\, dx$

We do a u-substitution with $u = nx, du = n\, dx$:

$$\begin{aligned}
\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\, dx &= \lim_{n\to\infty} \int_{na}^\infty \frac{du}{1+u^2} \\
&= \lim_{n\to\infty} [\arctan u]_{na}^\infty \\
&= \lim_{n\to\infty} \left(\tfrac{\pi}{2} - \arctan na\right) = \begin{cases} \frac{\pi}{2} & a = 0 \\ 0 & a > 0 \\ \pi & a < 0 \end{cases}.
\end{aligned}$$

8. a) Find the smallest constant c such that $\log(1+e^t) < c + t$ for $0 < t < \infty$.
First, observe that

$$1 + e^t < e^c e^t \quad \implies \quad \tfrac{1+e^t}{e^t} < e^c.$$

Note that

$$\lim_{t\to 0} \tfrac{1+e^t}{e^t} = 2 \quad \text{and} \quad \lim_{t\to\infty} \tfrac{1+e^t}{e^t} = 1.$$

Since $\frac{1+e^t}{e^t}$ is monotonic, it is evidently monotonically decreasing:

$$\tfrac{d}{dt}\tfrac{1+e^t}{e^t} = \tfrac{d}{dt}\left(\tfrac{1}{e^t}+1\right) = \tfrac{d}{dt}e^{-t} = -e^{-t}.$$

Thus, let $e^2 = 2$ or $c = \log 2$.

b) Does $\lim\limits_{n\to\infty} \frac{1}{n}\int_0^1 \log(1+e^{nf(x)})\,dx$ exist for every real $f \in L^1[0,1]$, if $f > 0$?

From part (a), we get $1 < \frac{1+e^{nf(x)}}{e^{nf(x)}} < 2$, which gives

$$e^{nf(x)} < 1 + e^{nf(x)} < 2e^{nf(x)}$$

$$nf(x) < \log\left(1+e^{nf(x)}\right) < nf(x) + c \qquad (c = \log 2)$$

$$n\int_0^1 f(x)\,dx < \int_0^1 \log\left(1+e^{nf(x)}\right)\,dx < n\int_0^1 f(x)\,dx + c$$

(since integration is a pos linear functional and $f \in L^1$.)

$$\int_0^1 f(x)\,dx < \frac{1}{n}\int_0^1 \log\left(1+e^{nf(x)}\right)\,dx < \int_0^1 f(x)\,dx + \frac{c}{n}$$

Then taking the limit as $n \to \infty$, we get

$$\int_0^1 f(x)\,dx \le \lim_{n\to\infty}\frac{1}{n}\int_0^1 \log\left(1+e^{nf(x)}\right)\,dx \le \int_0^1 f(x)\,dx.$$

Since f is integrable by hypothesis, the Sandwich Theorem gives

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1 \log\left(1+e^{nf(x)}\right)\,dx = \int_0^1 f(x)\,dx.$$

VI. The Hahn-Banach Theorem and applications

[Folland] It is not obvious that there are any nonzero bounded functionals on an arbitrary normed vector space. That such functionals exist in great abundance is one of the fundamental theorems of functional analysis.

[Reed & Simon] In dealing with Banach spaces, one often needs to construct linear functionals with certain properties. This is usually done in two steps: first one defines the linear functional on a subspace of the Banach space where it is easy to verify the desire properties; second, one appeals to (or proves) a general theorem which says that any such functional can be extended to the whole space while retaining the desired properties. One of the basic tools the second step is the following theorem,

Theorem VI.1. Let X be a vector space and $p : X \to \mathbb{R}$ such that

(i) $p(\alpha x) = \alpha p(x)$, $\forall \alpha \geq 0$, and

(ii) $p(x + y) \leq p(x) + p(y)$, $\forall x, y \in X$.

If S is a subspace of X and there is a linear functional $f : S \to \mathbb{R}$ such that $f(s) \leq p(s)$, $\forall s \in S$, then f may be extended to $F : X \to \mathbb{R}$ with $F(x) \leq p(x)$, $\forall x \in X$, with $F(s) = f(s)\, \forall s \in S$.

Proof. The idea of the proof is to first show that if $x \in X$ but $x \notin S$, then we can extend f to a functional having all the right properties on the space spanned by x and S. We then use a Zorn's Lemma / Hausdorff Maximality argument to show that this process can be continued to extend f to the whole space X.

(Sketch)

1. Consider the family

$$\mathcal{G} := \{g : D \to \mathbb{R} \,\vdots\, g \text{ is linear; } g(x) \leq p(x),\ \forall x \in D;\ g(s) = f(s),\ \forall s \in S\},$$

where D is any subspace of X which contains S. So $\mathcal{G}$ is roughly the collection of "all linear extensions of f which are bounded by p". Now $\mathcal{G}$ is a poset under

$$g_1 \prec g_2 \quad \iff \quad \text{Dom}(g_1) \subseteq \text{Dom}(g_2) \text{ and } g_2\Big|_{\text{Dom}(g_1)} = g_1.$$

2. Use Hausdorff maximality Principle (or Zorn) to get a maximal linearly ordered subset $\{g_\alpha\} \subseteq \mathcal{G}$ which contains f. Define F on the union of the domains of the $\{g_\alpha\}$ by $F(x) = g_\alpha(x)$ for $x \in \text{Dom}(g_\alpha)$.

3. Show that this makes F into a well-defined linear functional which extends f, and that F is maximal in that $F \prec G \implies F = G$.

4. Show F is defined on all of X using the fact that F is maximal. Do this by showing that a linear functional defined on a *proper* subspace has a *proper* extension. (Hence F must be defined on all of X or it wouldn't be maximal.)

□

Proposition VI.2. (Hausdorff Maximality Principle)
$(A, \prec)$ is a poset $\implies \exists B \subseteq A$ such that B is a maximal *linearly* ordered subset. I.e., if C is linearly ordered, then $B \prec C \prec A \implies C = B$ or $C = A$.

Of course, the HBT is also readily extendable to the complex case:

Theorem VI.3. Let X be a complex vector space and p a real-valued function defined on X satisfying

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \,\forall x, y \in X, \forall \alpha, \beta \in \mathbb{C} \text{ with } |\alpha| + |\beta| = 1.$$

If S is a subspace of X and there is a complex linear functional $f : S \to \mathbb{R}$ such that $|f(s)| \leq p(s)$, $\forall s \in S$, then f may be extended to $F : X \to \mathbb{R}$ with $|F(x)| \leq p(x)$, $\forall x \in X$, with $F(s) = f(s) \,\forall s \in S$.

VI.1. **Principle Applications of the HBT.**

Most often, $p(x)$ is taken to be the norm of the Banach space in question.

1. M is a closed subspace of X and $x \in X \backslash M \implies \exists f \in X^*$ such that $f(x) \neq 0, f\big|_M = 0$. In fact, if $\delta = \inf_{y \in M} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.

   Define f on $M + \mathbb{C}x$ by $f(y + \lambda x) = \lambda\delta$ for $y \in M, \lambda \in \mathbb{C}$. Then
   $$f(x) = f(0 + \cdot x) = 1 \cdot \delta = \delta$$
   but for $m \in M$,
   $$f(m) = f(m + 0) = 0 \cdot \delta = 0.$$
   For $\lambda \neq 0$, we have
   $$|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \cdot \|\lambda^{-1}y + x\| = \|y + \lambda x\|$$
   because $\delta = \inf \|y + x\| \leq \|\lambda^{-1}y + x\|$ (putting in λ^{-1} for y). Using $p(x) = \|x\|$, apply the HBT to extend f from $M + \mathbb{C}x$ to all of X.

2. If $x \neq 0, x \in X$, then $\exists f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

   $M = \{0\}$ is trivially a closed subspace, so apply (1) with $\delta = \|x\|$.

3. The bounded linear functionals on X separate points.

   If $x \neq y$, then (2) shows $\exists f \in X^*$ such that $f(x - y) \neq 0$. I.e., $f(x) \neq f(y)$. This result indicates that X^* is BIG.

4. If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$, so $\hat{x} \in X^{**}$. Then $\varphi : x \mapsto \hat{x}$ is a linear isometry from X into X^{**}.

   $$\hat{x}(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \hat{x}(\alpha f) + \hat{x}(\beta g),$$
   so $\hat{x}$ is linear. This verifies that $\hat{x} \in X^{**}$.
   For $\varphi(x) = \hat{x}$, $\varphi(ax + by)$ is defined by
   $$ax \hat{+} by(f) = f(ax + by) = af(x) + bf(y) = a\hat{x}(f) + b\hat{y}(f),$$

so $\varphi(ax+by) = \widehat{ax+by} = a\hat{x}+b\hat{y} = a\varphi(x)+b\varphi(y)$ shows that $\varphi : x \mapsto \hat{x}$ is linear. Finally,

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|$$

shows that

$$\|\hat{x}\| = \sup_{\|f\|\leq 1} \frac{|\hat{x}(f)|}{\|f\|} = \sup_{\|f\|\leq 1} \frac{|f(x)|}{\|f\|} \leq \sup_{\|f\|\leq 1} \frac{\|f\| \cdot \|x\|}{\|f\|} = \|x\|.$$

To get the reverse inequality, note that (2) provides a function f_0 for which $|\hat{x}(f_0)| = |f_0(x)| = \|x\|$ and $\|f\| = 1$. Then

$$\|\hat{x}\| = \sup_{\|f\|=1} |\hat{x}(f)| \geq |\hat{x}(f_0)| = \|x\|.$$

We saw this for Hilbert spaces, but this example is applicable to general Banach spaces and requires none of the Hilbert space machinery (orthonormal basis, projection theorem, etc.) as the HBT takes care of a lot.

VI.2. Corollaries to the HBT.

1. If X is a normed linear space, Y a subspace of X, and $f \in Y^*$, then there exists $F \in X^*$ extending f and satisfying $\|F\|_{X^*} = \|f\|_{Y^*}$.

   Proof. Apply HBT with $p(x) = \|f\|_{Y^*}\|x\|_{X^*}$. □

2. Let X be a Banach space. If X^* is separable, then X is separable.

   Proof. Let $\{f_n\}$ be a dense set in X^*. Choose $x_n \in X$, $\|x_n\| = 1$ so that

   $$|f_n(x_n)| \geq \|f_n\|/2.$$

   Let $\mathcal{D}$ be the set of all finite linear combinations of the $\|x_n\|$ with rational coefficients. Since $\mathcal{D}$ is countable, we just need to show that $\mathcal{D}$ is dense in X.

   If $\mathcal{D}$ is not dense in X, then there is a $y \in X \backslash \mathcal{D}$ and a linear functional $f \in X^*$ such that $f(y) \neq 0$ but $f(x) = 0$ for all $x \in \mathcal{D}$, by application (1).

Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ which converges to f. Then

$$\begin{aligned} \|f - f_{n_k}\| &\geq |(f - f_{n_k})(x_{n_k})| \\ &= |(f_{n_k})(x_{n_k})| \\ &\geq \|f_{n_k}\|/2 \end{aligned}$$

which implies $\|f_{n_k}\| \to 0$ as $k \to \infty$. Thus $f = 0$. $\lightning$

Therefore $\mathcal{D}$ is dense and X is separable. □

The example of ℓ_1 and ℓ_∞ shows that the converse of this theorem doesn't hold. In fact, this corollary offers a proof that ℓ_1 is not the dual of ℓ_∞, provided you can show ℓ_∞ is not separable.

VII. The Baire Theorem and Consequences

Definition VII.1. D is *dense* in X iff $\bar{D} = X$, equivalently, iff $\forall U$ open in X, $U \cap D \neq \varnothing$.

Definition VII.2. E is *nowhere dense* iff $\left(\bar{E}\right)^{\sim}$ is dense in X.

Definition VII.3. X is *meager*[10] iff X is a countable union of nowhere dense sets.

Theorem VII.4. Let (X, d) be a complete metric space. (Note: can substitute LCH for complete, using f.i.p. def of compactness). Then

a) If $\{O_n\}_{n=1}^{\infty}$ are open dense subsets of X, then $\bigcap_{n=1}^{\infty} O_n$ is dense in X.

b) No nonempty open subset of X is a countable union of nowhere dense sets. In particular, X is not.

Proof. The idea of the proof is straightforward: Suppose that X is a complete metric space and $X = \bigcup_{n=1}^{\infty} A_n$ with each A_n nowhere dense. We will construct a Cauchy sequence $\{x_m\}$ which stays away from each A_n so that its limit point x (which is in X by completeness) is in no A_n, thereby contradicting the statement $X = \bigcup_{n=1}^{\infty} A_n$.

Since A_1 is nowhere dense, we can find $x_1 \notin \bar{A}_1$ and an open ball B_1 about x_1 such that $B_1 \cap A_1 = \varnothing$, with the radius of B_1 smaller than 1.

Since A_2 is nowhere dense, we can find $x_2 \in B_1 \backslash \bar{A}_2$. Let B_2 be an open ball about x_2 such that $B_2 \cap A_2 = \varnothing$, with the radius of B_2 smaller than $\frac{1}{2}$.

Proceeding inductively, we obtain a sequence $\{x_n\}$ where $x_n \in B_{n-1} \backslash \bar{A}_n$, $\bar{B}_n \subseteq B_{n-1}$, and $B_n \cap A_n = \varnothing$.

This sequence is Cauchy because $n, m \geq N$ implies that for $x_n, x_m \in B_N$,

$$\rho(x_n, x_m) \leq 2^{1-N} + 2^{1-N} = 2^{2-N} \xrightarrow{N \to \infty} 0.$$

Let $x = \lim_{n \to \infty} x_n$. Since $x_n \in B_N$ for $n \geq N$, we have

$$x \in \bar{B}_N \subseteq B_{N-1}.$$

Thus $x \notin A_{N-1}$ for any N, which contradicts $X = \bigcup_{n=1}^{\infty} A_n$. $\square$

[10] Formerly called "of first category".

VII.1. **Applications of Baire.**

1. $\mathbb{Q}$ is NOT a G_δ.

   Proof. Suppose it were. Then $\mathbb{Q} = \bigcap_{n=1}^{\infty} O_n$, where the O_n are open.

   Note: $\mathbb{Q} \subseteq \bigcap O_n \implies \mathbb{Q} \subseteq O_n, \forall n$, so $\mathbb{Q}$ dense in $\mathbb{R} \implies O_n$ dense in $\mathbb{R}$, $\forall n$.

   Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q}$.
   Consider the singleton set (not the sequence!) $\{q_n\}$.

   Then $\{q_n\}$ closed $\implies \{q_n\}^{\sim}$ open. Also, $\{q_n\}^{\sim}$ is dense in $\mathbb{R}$. (It contains all but 1 of the rationals, so, e.g. the sequence $\{q_n - \frac{1}{m}\}_{m=1}^{\infty} \subseteq \{q_n\}^{\sim}$ and has q_n as a limit point.)

   Then $O_n \cap \{q_n\}^{\sim}$ is open and dense. To see dense, note that
   $$O_n \cap \{q_n\}^{\sim} = \mathbb{Q} \sim \{q_n\}.$$
   Then
   $$\bigcap (O_n \cap \{q_n\}^{\sim}) = (\bigcap O_n) \cap \left(\bigcap \{q_n\}^{\sim}\right) = \mathbb{Q} \cap (\mathbb{R} \sim \mathbb{Q}) = \varnothing,$$
   So $\varnothing$ is dense in $\mathbb{R}$, by Baire's Theorem. ↯ □

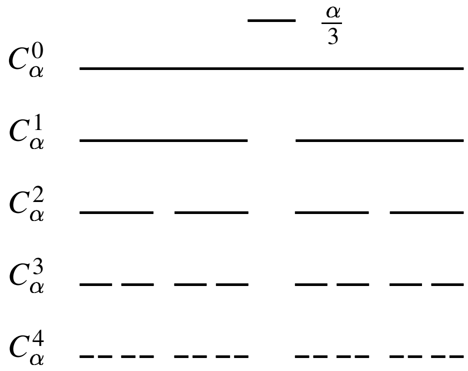
   However, $\mathbb{Q}$ is an F_σ: $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{q_n\}$.

2. A meager set with Lebesgue measure 1.
   We start by constructing a nowhere dense set with positive measure.
   Let $\mathcal{C}_\alpha$ be the Cantor set formed by removing an open interval of length $\frac{\alpha}{3^{n+1}}$ from each of the remaining 2^n pieces, where $0 < \alpha < 1$.

   Then $\mathcal{C}_\alpha = \bigcap_{n=0}^{\infty} \mathcal{C}_\alpha^n$ is closed as an intersection of closed sets and $\overline{\mathcal{C}_\alpha} = \mathcal{C}_\alpha$. To show $\mathcal{C}_\alpha$ is nowhere dense, it suffices to show that $\mathcal{C}_\alpha$ contains no nonempty open set.[11] If $O \neq \varnothing$ is open, then it contains

[11] If a closed set F contains no open set, then it is nowhere dense: we use contrapositive. Suppose $(\overline{F})^{\sim}$ is not dense. Then $\exists x \in F$ which is not a limit point of $\widetilde{F}$. If no sequence in $\widetilde{F}$ converges to x, every point of $\widetilde{F}$ must be at least $\varepsilon > 0$ away from x. Thus F contains $B_\varepsilon(x)$. The conditions are actually equivalent.

Figure 4. Construction of the Cantor set with measure α.



an interval A of positive length $\varepsilon > 0$. Each of the 2^n intervals of $\mathcal{C}_\alpha^n$ (n^{th} step of construction) are of length $\ell < 2^{-n}$, so for large enough N, we have $\ell < 2^{-N} < \varepsilon$. Thus A is longer than any component of $\mathcal{C}_\alpha^N$ and hence cannot be contained in $\mathcal{C}_\alpha^N$ or $\mathcal{C}_\alpha$. So $\mathcal{C}_\alpha$ is nowhere dense.

Now we use the measure lemma[12] to compute the measure of $\mathcal{C}_\alpha$:

$$\mu\mathcal{C}_\alpha = 1 - \sum_{n=0}^{\infty} \frac{\alpha 2^n}{3^{n+1}} = 1 - \frac{\alpha}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 - \frac{\alpha}{3}\left(\frac{1}{1-2/3}\right) = 1 - \alpha.$$

Thus, every singleton $\{\mathcal{C}_\alpha\}$ is a trivially a countable union of nowhere dense sets which have positive Lebesgue measure.

[12] The Measure Lemmae.

Proposition VII.5. Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) If $\{A_k\}$ is an increasing sequence ($A_k \subseteq A_{k+1}$) of sets of $\mathcal{A}$, then $\mu(\cup A_k) = \lim \mu A_k$.

b) If $\{A_k\}$ is a decreasing sequence ($A_{k+1} \subseteq A_k$) of sets of $\mathcal{A}$, and $\mu A_n < \infty$ for some n, then $\mu(\cap A_k) = \lim \mu A_k$.

($\mu A_n < \infty$ is necessary, else let $A_k = (k, \infty)$.)

Proposition VII.6. Let $(X, \mathcal{A})$ be a measurable space and let μ be a finitely additive function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$. Then μ is a measure if either

a) $\lim \mu A_k = \mu(\cup A_k)$ for each increasing sequence $\{A_k\} \subseteq \mathcal{A}$; or

b) $\lim \mu A_k = 0$ holds for each decreasing sequence $\{A_k\} \subseteq \mathcal{A}$ with $\bigcap A_k = \varnothing$.

However, to complete this in the manner suggested by Royden, we now consider $P = \bigcup_{k=1}^{\infty} \mathcal{C}_{1/k}$.

$$\mu P = \mu\left(\bigcup_{k=1}^{\infty} \mathcal{C}_{1/k}\right) = \lim_{k\to\infty}(1 - \tfrac{1}{k}) = 1.$$

So P is a union of a countable infinite collection of nowhere dense sets, and P has Lebesgue measure 1.

VII.2. Consequences of Baire.

Theorem VII.7 (Open Mapping Theorem)**.** X, Y are Banach spaces, $T \in L(X,Y)$. Then T surjective $\implies$ T open.

Proof. Nasty. (technical and longish) □

Corollary VII.8. If X, Y are Banach and $T \in L(X,Y)$ is bijective, then T is an isomorphism.

Proof.

$$\begin{aligned} T \text{ is an isomorphism} &\iff T^{-1} \in L(X,Y) \\ &\iff T^{-1} \text{ is continuous} \\ &\iff T \text{ is open} \end{aligned}$$

But this last condition is exactly the result of the OMT;

$$T \text{ bijective} \implies T \text{ surjective} \implies T \text{ open.}$$

□

Definition VII.9. The *graph of* T is the set

$$\Gamma(T) = \{(x,y) \in X \times Y : y = Tx\}.$$

Definition VII.10. $T \in L(X,Y)$ is *closed* iff $\Gamma(T)$ is a closed subspace of $X \times Y$.

Theorem VII.11 (Closed Graph Theorem)**.** If X, Y are Banach spaces, and the linear map $T : X \to Y$ is closed, then T is bounded.

Proof. Let π_1, π_2 be the projections of

$$\Gamma(T) = \{(x, y) \in X \times Y \vdots y = Tx\}$$

to X and Y respectively:

$$\pi_1(x, Tx) = x \qquad \text{and} \qquad \pi_2(x, Tx) = Tx.$$

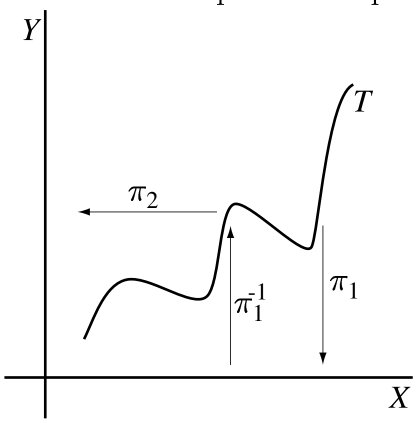
Obviously, $\pi_1 \in L(\Gamma(T), X)$ and $\pi_2 \in L(\Gamma(T), Y)$.

Since X, Y are complete, so is $X \times Y$. Hence T closed implies $\Gamma(T)$ is a closed subspace (by definition), and a closed subspace of a complete space is complete, so $\Gamma(T)$ is also complete.

Now $\pi_1 : \Gamma(T) \to X$ is a bijection, hence an isomorphism by the corollary to OMT, i.e., π_1^{-1} is bounded.

But then $T = \pi_2 \circ \pi_1^{-1}$ is bounded. □

FIGURE 5. T as a composition of projections



CGT restated: Let X, Y be Banach spaces, and $T : X \to Y$ be linear.

(1) Then T is bounded $\iff$ $\Gamma(T)$ is closed.

(2) Then T is continuous $\iff$ $(x_n \to x, Tx_n \to y \implies y = Tx)$.

Note. For X, Y be Banach spaces, and $S : X \to Y$ *unbounded* ,

(a) $\Gamma(S)$ is not complete,

(b) $T : X \to \Gamma(S)$ is closed but not bounded, and

(c) $T^{-1} : \Gamma(S) \to X$ is bounded and surjective but not open.

Theorem VII.12 (Uniform Boundedness Principle, aka Banach-Steinhaus Theoreom)**.** Suppose X, Y are normed vector spaces and and $\mathcal{A} \subseteq L(x, y)$.

a) If $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all x in some nonmeager set $D \subseteq X$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

b) If X is Banach and $\sup_{T\in\mathcal{A}} \|Tx\| < \infty \, \forall x \in X$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

Proof of (a). Let

$$E_n = \{x \in X : \sup_{T\in\mathcal{A}} \|Tx\| \leq n\} = \bigcap_{T\in\mathcal{A}} \{x \in X : \|Tx\| \leq n\}.$$

Thus the E_n are closed as intersections of preimages of closed sets under continuous maps. Since $\sup_{T\in\mathcal{A}} \|Tx\| \leq N$, $\forall x \in D$ by hypothesis, E_N contains a nontrivial closed ball

$$\overline{B(x_0, r)}, r > 0.$$

But then

$$\begin{aligned} \|x\| < R &\implies (x - x_0) \in E_N \\ &\implies \|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| \leq 2N, \end{aligned}$$

so $\|x\| \leq r \implies \|Tx\| \leq 2N \, \forall T \in \mathcal{A}$, which implies that

$$\|T\| \leq \tfrac{2N}{r} < \infty.$$

□

Proof of (b). X is a nonempty Banach space, so X is nonmeager by Baire. (Baire's Theorem says that every complete metric space is nonmeager.) Then just apply (a). □

Rephrase of the UBP:
Either $\exists M < \infty$ such that $\|T\| \leq M$, $\forall T \in \mathcal{A}$,
or else $\sup_{T\in\mathcal{A}} \|Tx\| = \infty$, $\forall x$ in some dense $G_\delta \subseteq X$.

Geometrically, either there is a ball $B \subseteq Y$ (with radius M, center 0) such that every $T \in \mathcal{A}$ maps the unit ball of X into B, *or* there exists an $x \in X$ (in fact, a whole dense G_δ of them) such that no ball in Y contains Tx, for all $T \in \mathcal{A}$ simultaneously.

VII.3. **Related Problems.**

1. Find a Banach space X, a normed linear space Y, and a continuous linear bijection $f : X \to Y$ such that f^{-1} is not continuous. (Note: Y better not be Banach!)

Let $X := \big(L^2[0,1], \|\cdot\|_2\big)$ and $Y := \big(L^2[0,1], \|\cdot\|_1\big)$, and define $f : X \to Y$ by $f(x) = x$.

f is clearly bijective and linear. To see f is continuous, it suffices to show f is continuous at 0:

Let $\varphi \in X$ and fix $\varepsilon > 0$. Since X, Y are finite measure spaces, $p < q \implies \|\varphi\|_p \le \|\varphi\|_q$.

Thus, $\|\varphi\|_1 \le \|\varphi\|_2$ shows $\|\varphi\|_2 \le \varepsilon \implies \|f(\varphi)\|_1 = \|\varphi\|_1 \le \varepsilon$.

To see f^{-1} is not continuous, consider $\{\varphi_n\} \subseteq Y$ defined by

$$\varphi_n(x) = \frac{1}{\sqrt{x + 1/n}}.$$

Now $\varphi_n(x) \le \varphi_{n+1}(x) \le \varphi(x) = \frac{1}{\sqrt{x}}$, $\forall n, x$. Since $\int_0^1 \frac{1}{\sqrt{x}} dx = 2 \implies \varphi \in L^1$, the MCT gives

$$\begin{aligned}\lim \|\varphi\|_1 = \lim \int_0^1 \frac{dx}{\sqrt{x+1/n}} &= \int_0^1 (\lim \varphi_n) dx \\ &= \int_0^1 \frac{dx}{\sqrt{x}} \\ &= 2 = \|\varphi\|_1.\end{aligned}$$

However,

$$\|\varphi_n\|_2 = \left(\int_0^1 \frac{dx}{x + 1/n}\right)^{1/2} = (\log[1+n])^{1/2} \xrightarrow{n\to\infty} \infty$$

shows that $\lim f^{-1}(\varphi_n)$ does not converge.

Indeed, $\int_0^1 \varphi^2 \, dx = \int_0^1 \frac{dx}{x} = \infty$. Hence, $\lim f^{-1}(\varphi_n) \neq f^{-1}(\varphi)$ shows that f^{-1} is not continuous.

2. Let V be a Banach space.

   a) If V is infinite-dimensional, construct an unbounded linear operator $f : V \to V$.

   b) If V is finite-dimensional, show that every linear operator on it is bounded.

Since V is a vector space, it has some basis $\{e_\lambda\}_{\lambda\in\Lambda}$. Then if $\{c_\lambda\}_{\lambda\in\Lambda}$ is any sequence (or net) in $\mathbb{R}$, $\exists! f : V \to V$ such that for every element $v = \sum v_\lambda e_\lambda$ of V, we have

$$f(v) = \sum c_\lambda v_\lambda e_\lambda.$$

(Just define $f(e_\lambda) = c_\lambda e_\lambda$.)

a) For V infinite-dimensional, $\{c_\lambda\}$ unbounded implies f unbounded.

b) For V finite-dimensional, $\|f\| \leq k \cdot \max\{c_\lambda\}$ for some k. (Basically, Λ finite $\implies$ $\max\{c_\lambda\}$ exists.)

See any book on Quantum Dynamics or Functional Analysis for the examples of the position & momentum operators (which are unbounded), e.g., Reed & Simon.