Exam 1

1. Use $\epsilon - \delta$ language to prove $\lim_{x\to 0} x^4 = 0$.

2. Determine whether each limit exists as a real number, as ∞ or $-\infty$, or fails to exist. If the limit exists, evaluate it.
a) $\lim_{x\to -2} \frac{|x^2+3x+2|}{x+2}$ b $\lim_{x\to 0} \frac{1-cos^2 x}{x}$
c) $\lim_{x\to 0} sin^2(x)cos(\frac{1}{x})$ d) $\lim_{x\to 2} \frac{sin(x-2)}{(x-2)^{3/2}}$

3. Find derivatives.
a)$\frac{x}{x^2+1}$ b) e^{x^2+1} c)$cos(2x)$.

4. Find the equation of the tangent line and normal line at the given point.
a).$f(x) = \sqrt{x+1}$ at $a = 1$
b).$f(x) = 2^x - ln2$ at $a = 0$

Solution to the sample exam 1

1. Want to find $\delta > 0$, such that if $0 < |x - 0| < \delta$, then $|x^4 - 0| < \epsilon$, or if $0 < |x| < \delta$, then $|x^4| < \epsilon$. We know $|x^4| < \epsilon \Rightarrow |x| < \epsilon^{1/4}$. So we let $\delta = \epsilon^{1/4}$.

2.a).$\lim_{x\to -2} \frac{|x^2+3x+2|}{x+2} = \lim_{x\to -2} \frac{|(x+2)(x+1)|}{x+2}$. In order to be able to remove the absolute value sign, we need to evaluate the one side limits:
If $x \to -2^-$, $(x+2)(x+1) > 0$, so $|(x+2)(x+1)| = (x+2)(x+1)$, therefore, $\lim_{x\to -2^-} \frac{|(x+2)(x+1)|}{x+2} = \lim_{x\to -2^-} \frac{(x+2)(x+1)}{x+2} = \lim_{x\to -2^-}(x+1) = -1$.
If$x \to -2^+$, $(x+2)(x+1) < 0$, so $|(x+2)(x+1)| = -(x+2)(x+1)$, therefore, $\lim_{x\to -2^+} \frac{|(x+2)(x+1)|}{x+2} = \lim_{x\to -2^+} \frac{-(x+2)(x+1)}{x+2} = \lim_{x\to -2^+} -(x+1) = 1$.Since the right hand and left hand limit do not agree, the limit does not exist.

b)$\lim_{x\to 0} \frac{1-cos^2x}{x} = \lim_{x\to 0} \frac{sin^2x}{x} = \lim_{x\to 0} \frac{sinx}{x} \cdot sinx = \lim_{x\to 0} \frac{sinx}{x} \lim_{x\to 0} sinx = 0$.

c) Using squeezing theorem: $|cos(\frac{1}{x})| \leq 1 \Rightarrow |sin^2xcos(\frac{1}{x})| \leq sin^2x$. $\Rightarrow -sin^2x \leq sin^2xcos(\frac{1}{x}) \leq sin^2x$. Since $\lim_{x\to 0} sin^2x = \lim_{x\to 0} -sin^2x = 0$, the middle term also has a limit 0.
d).Let y=x-2. Then $\lim_{x\to 2} \frac{sin(x-2)}{(x-2)^{3/2}} = \lim_{y\to 0} \frac{siny}{y^{3/2}} = \lim_{y\to 0} \frac{siny}{y\cdot y^{1/2}} = \lim_{y\to 0} \frac{siny}{y} \cdot \frac{1}{y^{1/2}} = \lim_{y\to 0} \frac{siny}{y} \lim_{y\to 0} \frac{1}{y^{1/2}}$ The first limit is 1. the second limit does not exist. So the whole limit does not exist.

3.a).$(\frac{x}{x^2+1})' = \frac{(x)'(x^2+1)-x(x^2+1)'}{(x^2+1)^2} = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$.
b).Use the formula $(e^{f(x)})' = e^{f(x)}f'(x)$. We have $(e^{x^2+1})' = e^{x^2+1} \cdot (x^2+1)' = e^{x^2+1} \cdot 2x$.
c).Newton's notation: Decompose $cos2x$ using $f(x) = 2x$ and $g(x) = cosx$. Then $f'(x) = 2$, $g'(x) = -sinx$, so $g'(f(x)) = -sin(2x)$. Then by the chain rule $(cos2x)' = -2sin(2x)$.
Leibniz's notation: The chain rule under this notation is $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. Now let $y = cos2x$ whose derivative we're about to calculate, let $u = 2x$ is the intermediate variable. Use u to substitute x, then y become a function of u: $y = cos(u)$. Then do the calculation, $\frac{dy}{du} = -sin(u)$, $\frac{du}{dx} = 2$, insert these two into the chain rule formula, we get $\frac{dy}{dx} = -sin(u) \cdot 2 = -2sin(2x)$.

4.a)$f(x) = \sqrt{x+1}$. Decompose it into the form $f(x) = g(h(x))$, where $g(x) = \sqrt{x}$, and $h(x) = x+1$. Calculate the derivatives for g and h: $g'(x) = \frac{1}{2\sqrt{x}}$, $h'(x) = 1$. So by the chain rule $f'(x) = g'(h(x))h'(x) = \frac{1}{2\sqrt{x+1}}$. Since here we have $a = 1$, then $f(a) = f(1) = \sqrt{2}$ and $f'(1) = \frac{1}{2\sqrt{2}}$. Insert to the equation of the tangent line: $y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x-1)$. The slope of the normal line is $-\frac{1}{f'(a)}$,so the equation of the normal line is $y - \sqrt{2} = -2\sqrt{2}(x-1)$. The slope is the only thing that is different in these two equations.
b).$f(x) = 2^x - ln2 \Rightarrow f'(x) = 2^x ln2$, so $f(0) = 1 - ln2$ and $f'(0) = ln2$. Then the tangent line is $y - (1 - ln2) = (ln2)x$, the normal line $y - (1 - ln2) = -\frac{1}{ln2}x$.