

## Solutions to Final Exam

1. An ant is heading toward a circular region of pancake batter. The ant cannot breathe while walking through the batter. The batter has a radius of 4 cm. The ant is going to walk from a point 6 cm due north of the batter's center to a point 5 cm due west of the batter's center.

If the ant moves at 0.8 cm/sec, how long will it have to hold its breath?

*Solution:*

We set up a coordinate system with the center of circle of batter as the origin. Then the line of the ant's path has slope  $\frac{6-0}{0-(-5)} = \frac{6}{5}$ . The equation of the line is thus

$$y = \frac{6}{5}x + 6.$$

The batter is represented by the circle with equation

$$x^2 + y^2 = 4^2 = 16.$$

We thus seek the intersection of this line and this circle:

$$x^2 + \left(\frac{6}{5}x + 6\right)^2 = 16$$

$$x^2 + \frac{36}{25}x^2 + \frac{72}{5}x + 36 = 16$$

$$\frac{61}{25}x^2 + \frac{72}{5}x + 20 = 0$$

$$x = \frac{-\frac{72}{5} \pm \sqrt{\left(\frac{72}{5}\right)^2 - 4\left(\frac{61}{25}\right)(20)}}{2 \cdot \frac{61}{25}} = -2.236236 \text{ and } -3.665393.$$

At these values of  $x$ , we have

$$y = 3.3165047 \text{ and } y = 1.601528$$

The distance travelled by the ant through the batter is thus 2.232399 cm, and the time that the ant has to hold its breath is

$$\frac{2.232399 \text{ cm}}{0.8 \text{ cm/sec}} = 2.790499 \text{ sec.}$$

2. The populations of the cities of Alexandria and Springfield are growing exponentially. In 1980, the population of Alexandria was 120,000 and the population of Springfield was 85,000. In 1995, the population of Alexandria was 185,000. The population of Springfield triples every 25 years.

In what year will there be twice as many people in Springfield as in Alexandria?

*Solution:*

Alexandria:

Say 1980 is  $t = 0$ . Then if  $A(t)$  is the population (in thousands) of Alexandria  $t$  years after 1980, we have

$$\begin{aligned} A(t) &= A_0 b^t \\ A(0) &= 120 = A_0 b^0 = A_0 \\ A(15) &= 185 = A_0 b^{15} = 120 b^{15} \\ b^{15} &= \frac{185}{120} \\ b &= \left(\frac{185}{120}\right)^{1/15} = 1.02927802. \end{aligned}$$

So  $A(t) = 120(1.02927802)^t$ .

Springfield:

With a similar setup, we have

$$\begin{aligned} S(t) &= C_0 d^t \\ S(0) &= 85 = C_0 d^0 = C_0 \\ S(25) &= 3 \cdot 85 = 85 d^{25} \\ 3 &= d^{25} \\ d &= 3^{1/25} = 1.04492435. \end{aligned}$$

So  $S(t) = 85(1.04492435)^t$ .

Now, the question asks when there will be twice as many people in Springfield as in Alexandria, so we need to solve the equation

$$\begin{aligned} S(t) &= 2A(t) \\ 85(1.04492435)^t &= 2 \cdot 120(1.02927802)^t \\ \ln 85 + t \ln 1.04492435 &= \ln 240 + t \ln 1.02927802 \\ t &= \frac{\ln 240 - \ln 85}{\ln 1.04492435 - \ln 1.02927802} = 68.8007 \end{aligned}$$

or the year 2048.

3. Suppose the value of my shoehorn collection is growing according to a linear model. In 2000, it was worth \$4.80. In 2002, it was worth \$5.03.

My toothpaste collection is also growing in value according to a linear model. In 1990, the collection was worth \$2.22, while in 1995, it was worth \$3.19.

When will my toothpaste collection be worth \$5.00 more than my shoehorn collection?

*Solution:*

Let the year 2000 be represented by  $t = 0$ . Then, for shoehorns we have the data points  $(0, 4.80)$ , and  $(2, 5.03)$ , so our linear model is

$$S(t) = 4.80 + 0.115t$$

For toothpaste, the data points are  $(-10, 2.22)$  and  $(-5, 3.19)$ , so the linear model is

$$T(t) = 4.16 + 0.194t$$

The problem thus asks us to solve the equation

$$T(t) = 5 + S(t)$$

from which we find

$$t = 71.3924$$

or the year 2071.

4. The amount of radiation from a star is observed to be a sinusoidal function of time. You measure the radiation output at its maximum of 400 megawatts at 3 AM one morning. It then drops to its minimum of 180 megawatts at 8 PM that evening. What percentage of the time is the radiation from the star greater than 350 megawatts?

*Solution:*

Using the standard sinusoidal model,

$$R(t) = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D$$

we have

$$A = \frac{400 - 180}{2} = 110$$

$$B = 2 \cdot 17 = 34$$

$$D = \frac{400 + 180}{2} = 290$$

$$C = 3 - \frac{B}{4} = -5.5$$

To answer the question we need first to solve  $R(t) = 350$ :

$$R(t) = 110 \sin\left(\frac{2\pi}{34}(t + 5.5)\right) + 290 = 350$$

$$\frac{2\pi}{34}(t + 5.5) = \sin^{-1} \frac{60}{110} = 0.5769313$$
$$t = -2.37807.$$

The symmetry solution is

$$3 + (3 - (-2.37807)) = 8.37807.$$

Hence, the time above 350 MW per period is  $8.37807 - (-2.37807) = 10.7561$  hours which is

$$\frac{10.7561}{34} = 31.636\%$$

of the time.

5. Susie starts running at a constant speed in a straight line from a point 20 meters due south of a light post in a park to a bench that is 40 meters due east of the light post. It will take her 15 seconds to get to the bench. Express her distance from the light post as a function of the time  $t$  since she started running.

*Solution:*

We set up a coordinate system with the light post at the origin, so that her starting point is  $(0, -20)$  and the bench is at  $(40, 0)$ . Since Susie is moving in a straight line at a constant speed, we know that the  $x$ - and  $y$ -coordinates of her position can be described as linear functions of time. So, for some constants  $a, b, c,$  and  $d,$  her location  $(x(t), y(t))$  at time  $t$  is given by

$$x(t) = a + bt, y(t) = c + dt.$$

We can take  $t$  to be the time since she starts, so that

$$x(0) = a = 0$$

and

$$y(0) = -20 = c$$

Also,

$$x(15) = 40 = 0 + 15b$$

so  $b = \frac{40}{15}$  and

$$y(15) = 0 = -20 + 15d$$

so  $d = \frac{20}{15}$ . Finally then,

$$x(t) = \frac{40}{15}t, \text{ and } y(t) = -20 + \frac{20}{15}t.$$

Thus her distance to the origin (light post) is given by

$$D(t) = \sqrt{\left(\frac{40}{15}t\right)^2 + \left(-20 + \frac{20}{15}t\right)^2}.$$

6. Find all values of  $d$  so that the quadratic function

$$f(x) = x(x + d) + 2d$$

has its vertex on the  $x$ -axis.

*Solution:*

We have

$$f(x) = x^2 + dx + 2d = \left(x + \frac{d}{2}\right)^2 - \frac{d^2}{4} + 2d$$

so the  $y$ -coordinate of the vertex is

$$y = 2d - \frac{d^2}{4}.$$

For the vertex to be on the  $x$ -axis, this must be zero so

$$0 = 2d - \frac{d^2}{4}$$

$$0 = d\left(2 - \frac{d}{4}\right)$$

so  $d = 0$  or  $d = 8$ .

7. Let  $g(x) = \frac{3x + 6}{5x - 7}$ .

Find  $g^{-1}(x)$ .

*Solution:*

Let  $y = g(x)$  and solve for  $x$ :

$$y = \frac{3x + 6}{5x - 7}$$

$$5yx - 7y = 3x + 6$$

$$(5y - 3)x = 6 + 7y$$

$$x = \frac{6 + 7y}{5y - 3}$$

so

$$g^{-1}(x) = \frac{6 + 7x}{5x - 3}.$$

8. Let  $f(x) = \frac{(x + 3)(2x - 5)}{x^2 - 5x - 6}$ .

- Find the zeros of  $f(x)$ .
- Find the horizontal asymptote of  $f(x)$  or state that it has none.
- Find all vertical asymptotes of  $f(x)$ .

*Solution:*

- (a) If  $f(x) = 0$ , then

$$(x + 3)(2x - 5) = 0$$

so  $x = -3$  or  $x = \frac{5}{2}$ . Since the denominator factors as

$$(x - 6)(x + 1)$$

we see that the denominator is not made zero by these values, so they are the zeros of  $f(x)$ .

- (b) We may write

$$f(x) = \frac{(x + 3)(2x - 5)}{x^2 - 5x - 6} = \frac{2x^2 + x - 15}{x^2 - 5x - 6} \left( \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \frac{2 + \frac{1}{x} - \frac{15}{x^2}}{1 - \frac{5}{x} - \frac{6}{x^2}} \approx \frac{2 + 0 - 0}{1 - 0 - 0} = 2$$

when  $x$  is large. Thus,  $y = 2$  is the horizontal asymptote.

- (c) Since the denominator of  $f(x)$  factors as

$$(x - 6)(x + 1)$$

we see that the function has vertical asymptotes at  $x = 6$  and  $x = -1$  (noting that the numerator is not made zero by these values).

9. Patrick is riding his bicycle around a circular track. The track has a radius of 50 meters.

- (a) If a ray from the center of the track to Patrick sweeps out area at the rate of 150 square meters per second, how fast is Patrick moving?

*Solution:*

There are many ways to solve this problem. One is to note that the area of the entire circle swept out by the ray is

$$\pi(50^2) = 7853.98163 \text{ square meters.}$$

Since the ray sweeps out 150 square meters per second, it will take

$$\frac{7853.98163 \text{ square meters}}{150 \text{ square meters per second}} = 52.359878 \text{ seconds}$$

to sweep out the whole circle. That is, Patrick does a lap of the track every 52.359878 seconds. Hence, his angular speed is

$$\frac{2\pi}{52.359878} = 0.12 \text{ radians per second}$$

and so his linear speed is

$$(0.12 \text{ radians per second})(50 \text{ meters}) = 6 \text{ meters per second.}$$

- (b) Patrick's coach is standing at the edge of the track. Patrick passes her while riding at a constant 11 meters per second. What is the straight-line distance between Patrick and his coach 12 seconds later?

*Solution:*

Setting the origin at the center of the track and the coach at the point  $(50, 0)$ , Patrick's position  $t$  seconds after passing the coach is given by

$$(50 \cos(\omega t), 50 \sin(\omega t))$$

where  $\omega$  is Patrick's angular speed. Since his linear speed is 11 meters per second, his angular speed can be found using the relation  $v = r\omega$ , i.e.,

$$\omega = \frac{v}{r} = \frac{11 \text{ m/sec}}{50 \text{ m}} = 0.22 \text{ radians per second.}$$

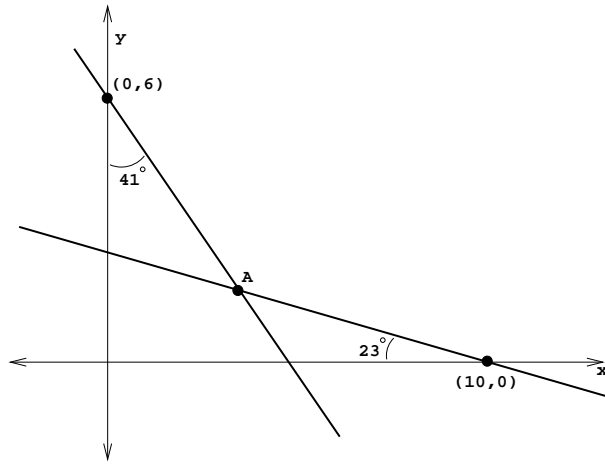
Thus, after 12 seconds, Patrick's location is

$$(50 \cos(0.22 \cdot 12), 50 \sin(0.22 \cdot 12)) = (-43.8409, 24.0411)$$

and so the distance from him to his coach is

$$\sqrt{(50 + 43.8409)^2 + 24.0411^2} = 96.8715 \text{ meters.}$$

10. Find the coordinates of the point labelled  $A$  in the figure below.



*Solution:*

The line passing through (0, 6) intersects the  $x$ -axis at a point with  $x$ -coordinate  $a$  with

$$\frac{a}{6} = \tan 41^\circ$$

i.e.,  $a = 5.2157204$ . Hence the line has slope

$$-\frac{6}{5.2157204} = -1.150368$$

and equation  $y = -1.150368x + 6$ .

The other line intersects the  $y$ -axis at the point (0,  $b$ ) with

$$\frac{b}{10} = \tan 23^\circ$$

so  $b = 10 \tan 23^\circ = 4.24474816$  and the line has equation

$$y = -0.424474816x + 4.24474816$$

Setting these two lines equal to each other and solving we find  $A$  has coordinates

$$x = 2.4180588, y = 3.218342.$$