Section 6.2 Verifying Identities.

Recall:

Pythagorean Identities:

$\sin^2 x + \cos^2 x = 1$; $\quad 1 + \cot^2 x = \csc^2 x$; $\quad \tan^2 x + 1 = \sec^2 x$

Ex: Prove the following identities.

1). $1 - \sec x \csc x \tan x = -\tan^2 x$

Techniques: ① Start from the more complicated side;

② convert all trig. functions to sine & cosine

$$LHS = 1 - \frac{1}{\cos x} \cdot \cancel{\frac{1}{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x} = 1 - \frac{1}{\cos^2 x}$$

$$= 1 - \sec^2 x$$

Since $\tan^2 x + 1 = \sec^2 x$

$1 - \sec^2 x = -\tan^2 x$

$$= -\tan^2 x$$

$$= RHS.$$

2) $\dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1+\sin\alpha}{\cos\alpha}$

Techniques: when proving $\frac{A}{B} = \frac{C}{D}$;

we can try to prove $\frac{A}{B} \cdot \left(\frac{C}{C}\right) = \frac{C}{D}$, or

$\frac{A}{B} \cdot \left(\frac{D}{D}\right) = \frac{C}{D}$.

$$LHS = \frac{\cos\alpha}{1-\sin\alpha} \cdot \frac{1+\sin\alpha}{1+\sin\alpha} = \frac{\cos\alpha(1+\sin\alpha)}{1-\sin^2\alpha} = \frac{\cancel{\cos\alpha}(1+\sin\alpha)}{\cos^{\cancel{2}}\alpha}$$

$$= \frac{1+\sin\alpha}{\cos\alpha} = RHS$$

(3) $\dfrac{\sec x - \cos x}{\cos x} = \tan^2 x$

Techique: convert $\dfrac{A-B}{C}$ into $\dfrac{A}{C} - \dfrac{B}{C}$

$$LHS = \frac{\sec x}{\cos x} - \frac{\cos x}{\cos x} = \frac{1}{\cos^2 x} - 1 = \sec^2 x - 1$$

$$= \tan^2 x = RHS.$$

since $1 + \tan^2 x = \sec^2 x$

Then $\sec^2 x - 1 = \tan^2 x$

(4) $2\tan^2 x = \dfrac{1}{\csc x - 1} - \dfrac{1}{\csc x + 1}$

Technique: find common denominator to combine the two rational expressions.

$$RHS = \frac{(\csc x + 1) - (\csc x - 1)}{(\csc x - 1)(\csc x + 1)} = \frac{2}{\csc^2 x - 1}$$

since $1 + \cot^2 x = \csc^2 x$

$\csc^2 x - 1 = \cot^2 x$

$$= \frac{2}{\cot^2 x} = \frac{2}{\frac{1}{\tan^2 x}}$$

$$= 2\tan^2 x = LHS.$$

(5) $(1 + \cot\alpha)^2 - 2\cot\alpha = \dfrac{1}{(1-\cos\alpha)(1+\cos\alpha)}$

Technique: work on both sides: $\left.\begin{array}{l} LHS = C \\ RHS = C \end{array}\right\} \Rightarrow LHS = RHS$

$$LHS = 1 + \cancel{2\cot\alpha} + \cot^2\alpha \cancel{-2\cot\alpha}$$

$$= \csc^2\alpha;$$

$$RHS = \frac{1}{1-\cos^2\alpha} = \frac{1}{\sin^2\alpha} = \csc^2\alpha;$$

$$LHS = RHS$$

Cofunction Identities:

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$; $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$; $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

$\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$; $\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$

The value of any trig function at an angle θ is equal to the value of its cofunction at $\left(\frac{\pi}{2}-\theta\right)$

ex: $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$; $\cos\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\cos\left(\frac{6\pi}{12}-\frac{4\pi}{12}\right)$

$=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$;

$\cos\pi=-1$; $\sin\left(\frac{\pi}{2}-\pi\right)=\sin\left(-\frac{\pi}{2}\right)=-1$

$\sec 122^\circ = \csc(90^\circ-122^\circ) = \csc(-32^\circ)$

$\tan 25^\circ = \cot(90^\circ-25^\circ) = \cot(65^\circ)$

Sum & Difference Identities:

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Applications of the sum & Difference I.D.:

If an angle θ is a sum or difference of two angles, α & β, and if we know the trig. function values of α & β, we can find the trig. function values of θ.

Ex: Evaluate the following.

1) $\cos(75^\circ) = \cos(45^\circ + 30^\circ)$
$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$
$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$

2) $\cos(\frac{\pi}{12}) = \cos(\frac{\pi}{3} - \frac{\pi}{4})$
$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

3) $\cos(\theta - \pi) = \cos\theta\cos\pi + \sin\theta\sin\pi$
$= -\cos\theta + 0 = \boxed{-\cos\theta}$

4) $\sin(\frac{5\pi}{12}) = \sin(\frac{\pi}{6} + \frac{\pi}{4})$
$= \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4}$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$ using sine of a sum ID

or $\sin(\frac{5\pi}{12}) = \cos(\frac{\pi}{2} - \frac{5\pi}{12}) = \cos(\frac{\pi}{12}) = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

using cofunction ID.

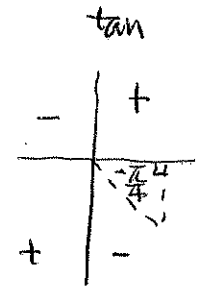
5) $\sin(-15^\circ) = \sin(30^\circ - 45^\circ)$
$= \sin 30^\circ \cos 45^\circ - \cos 30^\circ \sin 45^\circ$
$= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

6) $\sin(270^\circ - \theta) = \sin 270^\circ \cos\theta - \cos 270^\circ \sin\theta$
omit
$= -\cos\theta - 0 = \boxed{-\cos\theta}$

7). $\tan\left(-\frac{7\pi}{12}\right) = \tan\left(-\frac{3\pi}{12} - \frac{4\pi}{12}\right)$

$= \tan\left(-\frac{\pi}{4} - \frac{\pi}{3}\right)$

$= \dfrac{\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(-\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}$



$\tan\left(-\frac{\pi}{4}\right)$

$= -\tan\left(\frac{\pi}{4}\right)$

$= -\dfrac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = -1$

$= \dfrac{-1 - \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}}{1 + (-1)\cdot\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}}$

$= \dfrac{-1 - \frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{1}}{1 - \frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{1}}$

$= \dfrac{-1-\sqrt{3}}{1-\sqrt{3}} \cdot \dfrac{1+\sqrt{3}}{1+\sqrt{3}} = \dfrac{-1-\sqrt{3}-\sqrt{3}-3}{1-3}$

$= \dfrac{-4-2\sqrt{3}}{-2} = \dfrac{\cancel{-2}(2+\sqrt{3})}{\cancel{-2}} = \boxed{2+\sqrt{3}}$

Ex: Simplify each expression.

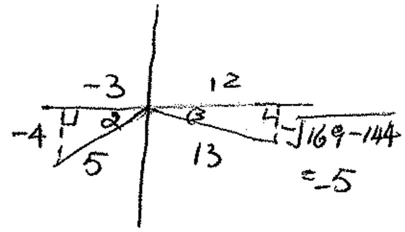
1) $\cos(47^\circ)\cos(2^\circ) + \sin(47^\circ)\sin(2^\circ)$

$= \cos(47^\circ - 2^\circ) = \cos 45^\circ = \boxed{\frac{\sqrt{2}}{2}}$

2) $\dfrac{\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{6}\right)}$

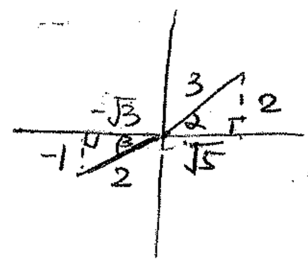
$= \tan\left(\frac{\pi}{12} - \frac{\pi}{6}\right) = \tan\left(\frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{4}\right) = \boxed{1}$

Ex: Find exact value of $\sin(\alpha-\beta)$, if $\sin\alpha = -\frac{4}{5}$, $\cos\beta = \frac{12}{13}$. α is in quadrant III, and β is quadrant IV.



$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= -\frac{4}{5}\cdot\frac{12}{13} - \frac{-3}{5}\cdot\frac{-5}{13}$$
$$= -\frac{48}{65} \quad -\frac{15}{65} = \boxed{-\frac{63}{65}}$$

Ex: Find the exact value of $\cos(\alpha+\beta)$, given $\sin\alpha = \frac{2}{3}$, $\sin\beta = -\frac{1}{2}$, α is in quadrant I, β is in quadrant III.



$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \frac{\sqrt{5}}{3}\cdot\frac{-\sqrt{3}}{2} - \frac{2}{3}\cdot\frac{-1}{2}$$
$$= \boxed{\frac{-\sqrt{15}+2}{6}}$$

§6.6 Conditional Trig. Equations

Terminologies:

Identities: equations that are always true.

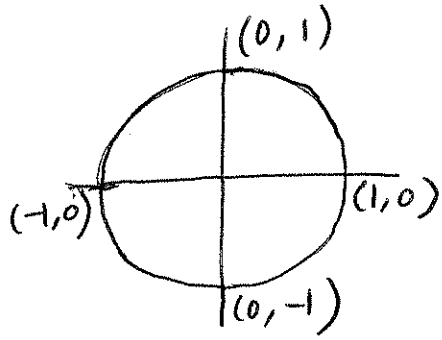
Conditional equations: not always true, but have at least one solution.

ex: $2x+1=5$

$2x=4$

$x=2$ (when $x=2$, the equation is true).

Solve Conditional Trig. Equations.



ex 1): Solve $\cos x = 0$

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$

Solution: $\{x \mid x = \frac{\pi}{2} + k\pi$, where k is any integer$\}$

ex 2). Solve $\cos x = 1$

$x = 0, \pm 2\pi, \pm 4\pi, \cdots$

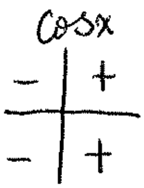
Solution: $\{x \mid x = 2k\pi$, where k is any integer$\}$

ex 3) Solve $\cos x = -1$

$x = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$

Solution: $\{x \mid x = \pi + 2k\pi$, where k is any integer$\}$

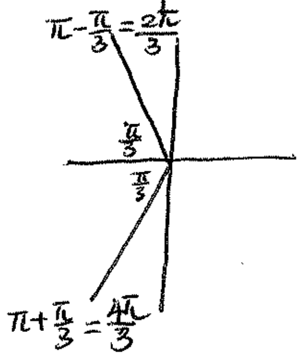
ex 4) Solve $\cos x = -\frac{1}{2}$.



Consider $\cos x = \frac{1}{2}$, $\Rightarrow$ the acute angle $x = \frac{\pi}{3}$

Therefore: $\cos x = -\frac{1}{2} \Rightarrow x$ can lie in quadrant II or III
with a reference angle $\frac{\pi}{3}$.

$\{x \mid x = \frac{2\pi}{3} + 2k\pi$ or $x = \frac{4\pi}{3} + 2k\pi$,
where k is any integer$\}$

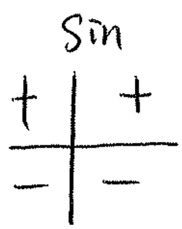


ex 5) solve $\sin x = -1$

$x = \pm\pi, \pm 3\pi, \pm 5\pi \cdots$

$\{x \mid x = \pi \pm 2k\pi$, where k is any integer$\}$.

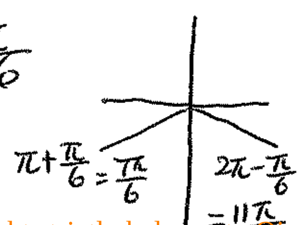
ex 6) solve $\sin x = -\frac{1}{2}$



Consider $\sin x = \frac{1}{2} \Rightarrow$ the acute angle $x = \frac{\pi}{6}$

Therefore, $\sin x = -\frac{1}{2} \Rightarrow x$ can lie in quadrant III IV
with an reference angle $\frac{\pi}{6}$

$\{x \mid x = \frac{7\pi}{6} + 2k\pi$ or $x = \frac{11\pi}{6} + 2k\pi$, where k is an integer$\}$



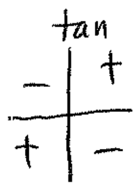
Key Concepts when solving $\cos x = a$ and $\sin x = a$:

1: $-1 \le \cos x \le 1$; $-1 \le \sin x \le 1$

2. If $a = \pm 1$ or 0, find the solution(s) on the x and/or y-axes.

3. If $-1 < a < 1$, and $a \neq 0$, x can lie in two different quadrants.

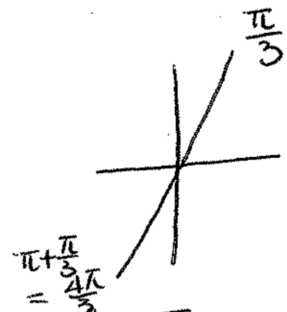
ex 7). solve $\tan \alpha = \sqrt{3}$



$$\frac{\sin \alpha}{\cos \alpha} = \sqrt{3} \Rightarrow \sin \alpha = \sqrt{3} \cos \alpha .$$

$$\frac{\sqrt{3}}{2} = \sqrt{3} \cdot \frac{1}{2} .$$

The acute angle $\alpha = \frac{\pi}{3}$.



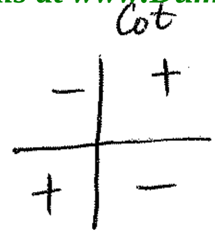
α can also lie in quadrant III with an ref. angle $\frac{\pi}{3}$, $\Rightarrow \alpha = \frac{4\pi}{3}$

$\{x \mid x = \frac{\pi}{3} + 2k\pi$, or $x = \frac{4\pi}{3} + 2k\pi$, for k is any integer$\}$

or

$\{x \mid x = \frac{\pi}{3} + k\pi$, where k is any integer$\}$.

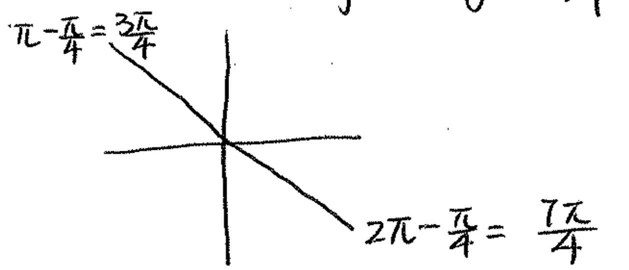
ex 8) solve $\cot \alpha = -1$



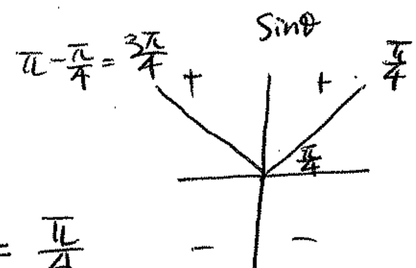
Consider $\cot \alpha = 1 \Rightarrow \frac{\cos\alpha}{\sin\alpha} = 1 \Rightarrow$ the acute angle $\alpha = \frac{\pi}{4}$.

Therefore $\cot\alpha = -1 \Rightarrow \alpha$ lies in quadrant II IV
with an ref. angle $\frac{\pi}{4}$

$\{\alpha \mid \alpha = \frac{3\pi}{4} + k\pi$, where
k is any integer$\}$



ex 9) Solve $\sin 2\alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.



Let $\theta = 2\alpha$
$\sin\theta = \frac{\sqrt{2}}{2} \Rightarrow$ the acute angle $\theta = \frac{\pi}{4}$
lie in quadrant II
with an ref. angle $\frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$.

$\{\theta \mid \theta = \frac{\pi}{4} + 2k\pi$ or $\theta = \frac{3\pi}{4} + 2k\pi$, where k is any integer$\}$

$\alpha = \frac{\theta}{2}$

$\{\alpha \mid \alpha = \frac{\pi}{8} + k\pi$ or $\alpha = \frac{3\pi}{8} + k\pi$, ...$\}$

ex 10) $\tan 3x = -\sqrt{3}$, where x is in the interval $(0, 2\pi)$

Let $\theta = 3x$

$\tan\theta = -\sqrt{3}$

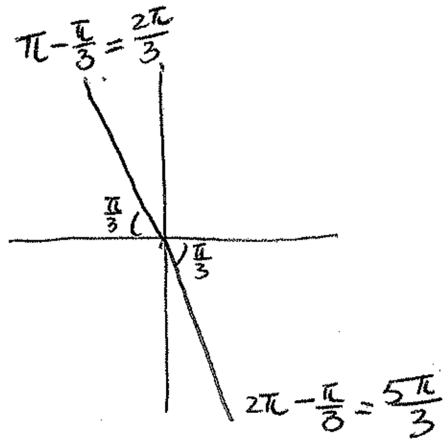
$$\tan\theta: \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

consider $\tan\theta = \sqrt{3} \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \sqrt{3} \Rightarrow \sin\theta = \sqrt{3}\cos\theta$

$\dfrac{\sqrt{3}}{2} = \sqrt{3} \cdot \dfrac{1}{2}$

The acute angle $\theta = \dfrac{\pi}{3}$.

Since $\tan\theta = -\sqrt{3}$, θ lies in quadrant II or IV, with an reference angle $\dfrac{\pi}{3}$



$\theta = \dfrac{2\pi}{3} + k\pi$,

$x = \dfrac{\theta}{3}$

$x = \dfrac{2\pi}{9} + \dfrac{k\pi}{3}$,

when $k=0$, $x = \dfrac{2\pi}{9}$

$k=1$, $x = \dfrac{2\pi}{9} + \dfrac{\pi}{3} = \dfrac{2\pi}{9} + \dfrac{3\pi}{9} = \dfrac{5\pi}{9}$

$k=2$, $x = \dfrac{2\pi}{9} + \dfrac{2\pi}{3} = \dfrac{2\pi}{9} + \dfrac{6\pi}{9} = \dfrac{8\pi}{9}$

$k=3$, $x = \dfrac{2\pi}{9} + \pi = \dfrac{2\pi}{9} + \dfrac{9\pi}{9} = \dfrac{11\pi}{9}$

$k=4$, $x = \dfrac{2\pi}{9} + \dfrac{4\pi}{3} = \dfrac{2\pi}{9} + \dfrac{12\pi}{9} = \dfrac{14\pi}{9}$

$k=5$, $x = \dfrac{2\pi}{9} + \dfrac{5\pi}{3} = \dfrac{2\pi}{9} + \dfrac{15\pi}{9} = \dfrac{17\pi}{9}$

$k=6$, $x = \dfrac{2\pi}{9} + \dfrac{6\pi}{3}$, outside of the interval $(0, 2\pi)$

So, $\left\{x \,\middle|\, x = \dfrac{2\pi}{9} + \dfrac{k\pi}{3}, \text{ for } k = 0, 1, 2, 3, 4, 5\right\}$.

or $\left\{x \,\middle|\, x = \dfrac{2\pi}{9}, \dfrac{5\pi}{9}, \dfrac{8\pi}{9}, \dfrac{11\pi}{9}, \dfrac{14\pi}{9}, \dfrac{17\pi}{9}\right\}$