Section 4.1 Exponential Functions and Their Applications

- Definition

An exponential function with base b is a function of the form

$$f(x) = b^x$$

where b and x are real numbers such that $b > 0$ and $b \neq 1$.

- Ex. Identify which of the following functions are exponential functions.

a) $f(x) = x^2$ b) $f(x) = 5^x$

c) $f(x) = (-2)^x$ d) $f(x) = \left(\frac{1}{3}\right)^{2x-1}$

Evaluate Exponential Functions

- Ex. Let $g(x) = 5^{2-x}$, and $h(x) = \left(\frac{2}{3}\right)^x$. Find the following values.

  a) $g(\frac{3}{2})$

  b) $h(-4)$

Graphs of Exponential Functions $f(x) = b^x$

- **Properties of Exponential Functions:** The exponential function $f(x) = b^x$ has the following properties:
  - 1. f is increasing for $b > 1$ and decreasing for $0 < b < 1$.
  - 2. The y-intercept is $(0, 1)$.
  - 3. The x-axis $(y = 0)$ is a horizontal asymptote.
  - 4. The domain of f is $(-\infty, \infty)$, and the range of f is $(0, \infty)$.

Ex. Sketch the graphs of $f(x) = 2^x$ and $f(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$.

Transformation of the Graph of an Exponential Function

The graph of a function of the form $g(x) = a \cdot b^{x-h} + k$ is a transformation of the graph of the function $f(x) = b^x$.

If:	The graph of $f(x)$:
$a > 1$	stretches
$0 < a < 1$	shrinks
$a < 0$	reflects around the x-axis
$h > 0$	moves to the right by h units
$h < 0$	moves to the left by $\|h\|$ units
$k > 0$	moves up by k units
$k < 0$	moves down by k units

- Ex. Sketch each of the following graph, and and find its domain, range, and horizontal asymptote.

  a) $y = 3^{x+2} - 4$ b) $y = -1 - 3^{-x}$

Exponential Equations

- For $b > 0$ and $b \neq 1$,

$$\text{if } b^{x_1} = b^{x_2}, \text{ then } x_1 = x_2.$$

- Ex. Solve each exponential equation.

  a) $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

  b) $\left(\frac{1}{10}\right)^x = 0.1$

  c) $2^{-x} = \frac{1}{8}$

  d) $\left(\frac{3}{5}\right)^x = \frac{125}{27}$

- the irrational number $e \approx 2.71$ is sometimes used in exponential functions as a base in real-life applications.

Section 4.3 Rules of logarithmic functions

Recall:

- The inverse rules of exponential and logarithmic equation:

  $x = a^y \Leftrightarrow y = log_a x$ where $a > 0$ and $a \neq 1$, and $x > 0$.

- Common Logarithm: $logx = log_{10} x$.

- Natural Logarithm: $lnx = log_e x$.

Properties of logarithms that can be derived from the inverse rule:

- $log_a 1 = 0,$ and $log_a a = 1$

  Therefore:

  - $log 1 = 0,$ and $log 10 = 1$
  - $ln 1 = 0,$ and $lne = 1$

More properties of logarithms

- Power Rule: $log_a M^p = p \cdot log_a M$
- Product Rule: $log_a MN = log_a M + log_a N$
- Quotient Rule: $log_a \frac{M}{N} = log_a M - log_a N$
- Other properties:

$$log_a a^x = x$$

$$a^{log_a x} = x$$

- Ex.1.Expresse each of the following as a product.

  a) lnx^6 b) $log_a\sqrt[4]{7}$ c) $log\frac{1}{3}$

- Ex.2. Express the following as a sum or difference of logarithms, and simplify if possible.

  a) $log_3(9 \cdot 27)$ b)$log_t\frac{8}{w}$ c) $ln\frac{x^2y^5}{z^4}$

  d) $log\sqrt{\frac{a^2b}{c^5}}$ e) $log_5 5t^5$

- Ex.3. Express the following as a single logarithm:

  a)$log_2 p^3 + log_2 q$ b)$\frac{1}{2}log(y) - \frac{1}{3}log(z)$

  c)$ln(x-1) + ln(3) - 3ln(x)$

- Ex.1. Simplify each of the following.

  a) $log_a a^8$ b) lne^{-t} c)$log10^{3k}$

- Ex.2. Simplify each of the following.

  a) $4^{log_4 k}$ b)e^{ln5} 10^{log7t}

Section 4.4 Solving Exponential and Logarithmic Equations

Definition

- Equations with variables in the exponents, such as $2^{5x} = 64$, are called exponential equations.
- Equations with variables in logarithms, such as $log_2 x + log_2(3x - 6) = 4$, are called logarithmic equations.

Case 1

If solving an equation involves a single logarithm or a single exponential expression, use the inverse relation between logarithmic and exponential equations:

$$y = log_a x \iff x = a^y$$

- Ex. Solve the following equation.

  $log_3(x+5) = -2$

Case 2

Use the following properties if possible:

- For any $a > 0$, $a \neq 1$,

$$a^x = a^y \Longleftrightarrow x = y$$

- For any $x > 0, y > 0, a > 0$ and $a \neq 1$,

$$log_a x = log_a y \Longleftrightarrow x = y$$

- Ex.1. Solve $2^{3x-7} = 32$.

  Solution: $2^{3x-7} = 2^5 \quad \Rightarrow \quad 3x - 7 = 5 \quad \Rightarrow \quad x = 4.$

- Ex.2. Solve $log_2(x+5) = log_2 10$.

  Solution: $x + 5 = 10 \quad \Rightarrow \quad x = 5.$

- Ex.3. Solve $4^{x^2-3x-13} = \frac{1}{64}$

Case 3

If an equation has only exponential expressions with different bases on each side, then take the natural logarithm or common logarithm of each side and use the power rule:

$$a^M = b^N \iff lna^M = lnb^N \iff M \cdot lna = N \cdot lnb$$

or

$$a^M = b^N \iff loga^M = logb^N \iff M \cdot loga = N \cdot logb$$

- Ex. Solve the following equations.

  a) $4^{x+3} = 3^{-x}$ b)$3^{2x-1} = 5^x$

Case 4

If an equation has several logarithms with the same base, apply a combination of rules stated in case 1-3, and the power, product and quotient rules:

$$log_a M^p = p \cdot log_a M$$

$$log_a MN = log_a M + log_a N$$

$$log_a \frac{M}{N} = log_a M - log_a N$$

Ex. Solve the following equations.

a) $log x + log(x+3) = 1$ b) $log_3(2x-1) - log_3(x-4) = 2$

c) $2ln(x) = ln(x+3) - ln(2)$

Note: You must check your answer when solving logarithmic equations!

Because you may obtain answers outside of the domain!

Solve $[log_5(z+2)]^2 = log_5(z+2)^2$

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- For any $x > 0, y > 0, a > 0$ and $a \neq 1$,

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- Ex.2. Solve $[log_5(z+2)]^2 = log_5(z+2)^2$

Case 2

Use the following properties if possible:

- For any $x > 0, y > 0, a > 0$ and $a \neq 1$,

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