Section 3.2 and 3.3 Zeros of Polynomial Functions

- A polynomial function is a function defined by a polynomial.
- For example, Linear functions are polynomial functions with degree 1, and quadratic functions are polynomial functions with degree 2.

Polynomial divisions:

- Dividend = Divisor · Quotient+Remainder.
- Synthetic division can only be used when dividing a polynomial by a binomial of the form $x - c$.
- Ex. Use synthetic division to find the quotient and remainder when $x^4 - 14x^2 + 5x - 9$ is divided by $x + 4$.

Zero of Polynomial Functions

Definition

If $y = P(x)$ is a polynomial function, then a value of x that satisfies $P(x) = 0$ is called a zero of the polynomial function.

- To find the zeros of a polynomial function $P(x)$, is equivalent to solve the equation $P(x) = 0$.

Ex.Find the zeros for $f(x) = 4x - 9$ and $g(x) = x^2 - 4x - 5$.

Note, in order to find zeros for higher ordered polynomial functions, we need to learn more properties and theorem about polynomial functions.

The Factor Theorem

Definition

- The number c is a zero of the polynomial function $y = P(x)$, if and only if $x - c$ is a factor of $P(x)$.

- Every polynomial function f of degree n, with $n \geq 1$, can be factored into n linear factors (not necessarily unique):
  $f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$,
  where c_1, c_2,..., c_n are zeros of $f(x)$.

- Ex.1. $f(x) = x^5 - x^4 - 9x^3 - 11x^2 - 4x = x(x+1)^3(x-4)$, therefore, the zeros of the $f(x)$ are 0, -1 and 4.

- Ex.2. Use synthetic division, determine whether $x + 4$ is a factor of the polynomial $P(x) = x^3 - 13x + 12$. If it is a factor, then factor $P(x)$ completely.

Definition

Multiplicity: If the factor $x - c$ occurs k times in the complete factorization of the polynomial $P(x)$, then c is a zero (root) of the polynomial with multiplicity k.

Ex.Identify the multiplicity of each zero in the previous examples.

Definition

n-Root Theorem:Every polynomial function of degree n has exact n zeros (counted with their multiplicity) in the system of complex numbers.

- For example: $f(x) = 4x - 9$ has one zero, $g(x) = x^2 - 4x - 5$ has two zeros, $f(x) = x^5 - x^4 - 9x^3 - 11x^2 - 4x$ has 5 zeros.

the Rational Zero Theorem

Definition

The Rational Zero Theorem: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$, where all the coefficients are integers.

Consider a rational number denoted by $\frac{p}{q}$, where p and q are relatively prime (in lowest terms). If $\frac{p}{q}$ is a zero of $P(x)$, then p is a factor of a_0 and q is a factor of a_n.

- Ex.1. Given $f(x) = 3x^4 - 11x^3 + 10x - 4$, find the rational zeros and then the other zeros.
- Ex.2. Given $f(x) = 2x^5 - x^4 - 4x^3 + 2x^2 - 30x + 15$, Find all zeros of the polynomial.

From the previous two examples, we observe the following:

Non-real zeros:

- If a complex number $a + bi$, $b \neq 0$, is a zero of a polynomial function $f(x)$ with real coefficients, then its conjugate, $a - bi$, is also a zero.
- Ex. If $2 + 7i$ is a zero of a polynomial function $f(x)$ with real coefficients, then its conjugate, $2 - 7i$, is also a zero of $f(x)$.

Irrational Zeros:

- If $a + c\sqrt{b}$, where a,b, and c are rational and b is not a perfect square, is a zero of a polynomial function $f(x)$ with rational coefficients, then its conjugate, $a - c\sqrt{b}$, is also a zero.
- Ex. If $-3 + 5\sqrt{2}$ is a zero of a polynomial function $f(x)$ with rational coefficients, then its conjugate, $-3 - 5\sqrt{2}$, is also a zero of $f(x)$.

3.2 and 3.3 Zeros of Polynomial Functions

Recall: A polynomial function is a function defined by a polynomial. For example, Linear functions are polynomial functions with degree 1, and quadratic functions are polynomial functions with degree 2.

Definition 1. For a polynomial function $P(x)$, a value of x that satisfies $P(x) = 0$ is called a zero of the polynomial function.

Thus, to find the zeros of a polynomial function $P(x)$, is equivalent to solve the equation $P(x) = 0$.

We have learned how to find the zeros for linear and quadratic functions, for example, we know how to solve $4x-9 = 0$ and $x^2-4x-5 = 0$ for x. In this class, we'll learn to find zeros for higher order (with a degree 3 or higher) polynomial functions , such as $f(x) = 3x^3 + 5x^2 - 7x + 2$ or $f(x) = 7x^5 - 3x^4 - 6x^2 + 9 - 13$.

Recall: When we find the zeros of a quadratic function such as $f(x) = x^2 - 4x - 5$, We could factor it so that

$$f(x) = x^2 - 4x - 5 = (x - 5)(x + 1) = 0.$$

Notice when the polynomial is factored, the equation become very easy to solve. In this case, $x = 5$ or $x = -1$. In other words, the zeros of the function $f(x)$ are 5 and -1.

In fact,

Theorem 2. *every polynomial function f of degree n, with $n \geq 1$, can be factored into n linear factors (not necessarily unique):*

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n),$$

where c_1, c_2,..., c_n are zeros of $f(x)$, and a_n is a constant.

For example,

$$f(x) = x^5 - x^4 - 9x^3 - 11x^2 - 4x = x(x + 1)^3(x - 4),$$

therefore, the zeros of the $f(x)$ are 0, -1 and 4. (We'll show you the technique to factor such polynomial in a few minutes! But first, let's take a look at some important properties about zeros of polynomial functions.)

Definition 3. Multiplicity: If the factor $x - c$ occurs k times in the complete factorization of the polynomial $P(x)$, then c is a zero (root) of the polynomial with multiplicity k.

Ex. Identify the multiplicity of each zero in the following examples:

1) $f(x) = x^2 - 2x + 1 = (x-1)^2 = 0$

2) $f(x) = x^5 - x^4 - 9x^3 - 11x^2 - 4x = x(x+1)^3(x-4)$.

Theorem 4. *n-Root Theorem: Every polynomial function of degree n has exact n zeros (counted with their multiplicity) in the system of complex numbers.*

For example:

$f(x) = 4x - 9$ has one zero,

$f(x) = x^2 - 4x - 5$ has two zeros,

$f(x) = x^5 - x^4 - 9x^3 - 11x^2 - 4x$ has 5 zeros.

From the above examples, we see that when a polynomial is factored, it's easy to find the zeros of that polynomial function. However, to factor a polynomial may be difficult. As a prerequisite, we must begin to learn how to divide polynomials using synthetic division.

Note that synthetic division can only be used when dividing a polynomial by a binomial of the form $x - c$.

Ex. Use synthetic division to find the quotient and remainder when $x^4 - 14x^2 + 5x - 9$ is divided by $x + 4$.

Note:

1) Dividend = Divisor $\cdot$ Quotient+Remainder;

2) The quotient is one degree less than the degree of the dividend.

3) If the remainder of a polynomial division $\frac{P(x)}{x-c}$ is 0, then $x - c$ is a factor of $P(x)$, and thus c is a zero of $P(x)$.

Theorem 5. *Rational Zero Theorem: Let* $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$*, where all the coefficients are integers. Consider a rational number denoted by* $\frac{p}{q}$*, where* p *and* q *are in lowest terms. If* $\frac{p}{q}$ *is a zero of* $P(x)$*, then* p *is a factor of* a_0 *and* q *is a factor of* a_n*.*

Ex. Given $f(x) = x^3 - 5x^2 + 11x + 17$, find the rational zeros and then the other zeros.

Ex. Given $f(x) = 3x^4 - 11x^3 + 10x - 4$, find all zeros of the polynomial.

Other theorems about zeros of polynomial functions:

Theorem 6. *If a complex number $a+bi$, $b \neq 0$, is a zero of a polynomial function $f(x)$ with real coefficients, then its conjugate, $a-bi$, is also a zero.*

Ex. If $2+7i$ is a zero of a polynomial function $f(x)$ with real coefficients, then its conjugate, $2-7i$, is also a zero of $f(x)$.

Theorem 7. *If an irrational number $a+c\sqrt{b}$ is a zero of a polynomial function $f(x)$ with rational coefficients, then its conjugate, $a-c\sqrt{b}$, is also a zero.*

Ex. If $-3+5\sqrt{2}$ is a zero of a polynomial function $f(x)$ with rational coefficients, then its conjugate, $-3-5\sqrt{2}$, is also a zero of $f(x)$.

Ex. If 0, $1-2i$, $2+\sqrt{5}$ are some zeros of a polynomial function $P(x)$ with rational coefficients, what is the lowest degree of $P(x)$?

Since complex conjugates and irrational conjugates show up in pairs, $1+2i$ and $2-\sqrt{5}$ must also be zeros of $P(x)$. Therefore, $P(x)$ has at least 5 zeros, and $P(x)$ has at least a degree of 5.

3.4 More Equation Solving

I. Solving Rational Equations

- Multiplying both sides of the equation by the least common denominator (LCD) to clear away fractions:

Ex. Solve the rational equations, and check your solutions.

- 1. $\frac{3x}{x+2} + \frac{6}{x} = \frac{12}{x^2+2x}$.
- 2. $\frac{1}{(5x-1)^2} + \frac{1}{5x-1} - 12 = 0$

Note: when solving a rational equation, you must check if your solutions are in the domain! Solution(s) outside the domain are incorrect!

II. Solving Radical Equations

- First isolate the radical on one side of the equation. If there are two radicals in the same equation, put one radical on each side.
- Use the principle of powers to clear away radicals:
  For any positive integer n, If $a = b$ is true, then $a^n = b^n$ is true .
  (Make sure to raise a power on each SIDE of the equation, NOT on each TERM of the equation.)

Ex. Solve the following radical equations, and check your solutions.

1) $5 + \sqrt{x+7} = x$ 2) $\sqrt{x-3} + \sqrt{x+5} = 4$

Note:

- When solving a radical equation, you must CHECK YOUR SOLUTIONS! Incorrect solution(s) may appear after raising an even power to each side of an equation.

III. Solve Equations with Absolute Value

Definition

The absolute value of a number x, denoted as $|x|$, is its distance from 0 on the number line.

For $a > 0$,

$$|x| = a \text{ is equivalent to } x = a \text{ or } x = -a,$$

and therefore,

$$|x| \geq 0$$

- Ex. Solve the absolute value equations, check your solution.

  1) $9 - |2x - 3| = 2$ 2) $|x^2 - 6| = 5x$ 3) $|a - 1| = |2a - 3|$

Solving equations using a mixture of techniques

- Ex.1. Solve $x^3 - x^2 - 5x + 5 = 0$ by grouping and factoring.
- Ex.2. Solve the following equations using substitution, then factoring or the quadratic formula.

  a) $t^4 - 81 = 0$ b) $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 10 = 0$

  c) $(v^2 - 4v)^2 - 17(v^2 - 4v) + 60 = 0$

  d) $x^4 - 12x^2 + 27 = 0$

Section 3.5. Graph of Polynomial Functions and Solve Polynomial Inequalities

Properties:

- A n-th degree polynomial function $P(x)$ has:
  - at most n real zeros, and thus at most n x-intercepts on its graph;
  - at most $n-1$ turning points (relative maximum or minimum) on its graph.
- Note: Only real-number zeros of a function correspond to the x-intercepts on its graph.

Ex. Use the graphs of linear and quadratic functions to verify the above properties.

More Properties of Polynomial Functions

- For a polynomial function $P(x)$,
  - if c is a real zero of $P(x)$ with even multiplicity, then the graph is tangent to the x-intercept $(c, 0)$.
  - if c is a real zero of $P(x)$ with odd multiplicity, then the graph crosses the x-intercept $(c, 0)$;
  - Ex. Consider the graphs of the quadratic functions below.

    a) $f(x) = x^2 + 8x + 16 = (x + 4)^2$

    b) $g(x) = x^2 - x - 12 = (x + 3)(x - 4)$.

- Show the Leading-Coefficient Test:(Page.320-321).

Guidelines to sketch a higher order polynomial in the form

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$:

- 1. Factor the polynomial.
- 2. Find x-intercept(s) by finding real zeros of the function and their multiplicities.
- 3. Find y-intercept: $(0, a_0)$.
- 4. Using leading term $a_n x^n$ to determine the end-behaviors of the graph.
- 5. Sketch the graph.

Ex.1. Sketch the graph of $f(x) = x^3 - 5x^2 + 7x - 3$.

Ex.2. Sketch the graph of $f(x) = -x^4 + 4x^2$.

Ex.3. Sketch the graph of $f(x) = -x^3 + x^2 + x - 1$

Section 3.6. Rational Functions

Definition

- A rational function is a quotient of two polynomials, that is,

$$f(x) = \frac{p(x)}{q(x)}, \quad \text{where } q(x) \neq 0.$$

- The domain of f consists of all x values for which $q(x) \neq 0$.

Vertical Asymptotes and Holes of a Rational Function $f(x) = \frac{p(x)}{q(x)}$

- When $f(x)$ is in its lowest terms, and c is a zero of the denominator, then the line $x = c$ is a vertical asymptote (V.A) on the graph of the function $f(x)$.
- If b is a zero of the denominator of $f(x)$ before $f(x)$ is simplified to its lowest terms, but b is NOT a zero of the denominator after $f(x)$ is simplified to its lowest terms, then there is a hole on the graph of $f(x)$ at $x = b$.

Ex. Find the vertical asymptote(s) and hole(s) of each rational function, and state the domain of the function.

a) $f(x) = \frac{1}{x}$ b) $f(x) = \frac{3x+5}{2x-6}$ c) $f(x) = \frac{2x-1}{x^2+2x-8}$ d)$f(x) = \frac{x-2}{x^2-4}$

Note: A rational function may have more than one V.A..

Horizontal Asymptotes

- A rational function $f(x)$ may have a horizontal asymptote (H.A.), $y = b$, if $f(x) \to b$ as $x \to \infty$ and/or as $x \to -\infty$.
- The H.A. of a rational function $f(x) = \frac{p(x)}{q(x)}$ depends on the degrees of the function's numerator and denominator.

degree of $p(x)$ and $q(x)$	**equation of the H.A.**
$deg.p(x) < deg.q(x)$	$y = 0$ (the x-axis)
$deg.p(x) = deg.q(x)$	$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$
$deg.p(x) > deg.q(x)$	no H.A

Note: A rational function has at most one H.A..

Ex. Find the horizontal asymptote(s) of each rational functions.

1) $f(x) = \frac{1}{x}$ b) $f(x) = \frac{3x+5}{2x-6}$ c) $f(x) = \frac{2x^2-11}{x^2+4x-7}$ d) $f(x) = \frac{x-3}{x^2-9}$

Note:

- The graph of a rational function $f(x)$ approaches its asymptote(s). It can never crosses its vertical asymptote(s), but it may cross the horizontal asymptote.

Ex. Graph the following rational functions. Make sure graph all asymptotes as dashed lines, label x-intercepts, y-intercepts, and some other points that can help to graph the functions.

a) $f(x) = \frac{2x+1}{x+3}$ b) $f(x) = \frac{3}{x^2-1}$ c) $f(x) = \frac{x-2}{x^2-4}$

Note:

- if $\frac{p(x)}{q(x)} = 0$, then $p(x) = 0$.
- Using even/odd function property may help you graph.

Section 3.6 RATIONAL FUNCTIONS

Definition 1. A rational function is a quotient of two polynomials, that is,

$$f(x) = \frac{p(x)}{q(x)}, \quad \text{where } q(x) \neq 0.$$

The domain of f consists of all x values for which $q(x) \neq 0$.

WHAT IS A VERTICAL ASYMPTOTE?

A vertical asymptote (V.A.) is a vertical line that the graph of a function cannot touch or cross. In fact, the graph of the function approaches ∞ or $-\infty$ near a vertical asymptote.

WHEN DOES A VERTICAL ASYMPTOTE OCCUR?

When a rational function, $f(x) = \frac{p(x)}{q(x)}$ is in its lowest terms (reduced), and c is a zero of the denominator (the value c makes the denominator equal to 0), then the vertical line $x = c$ is a vertical asymptote on the graph of the function $f(x)$.

Note: A rational function may have more than one vertical asymptote.

Ex. Find the vertical asymptote(s) of each rational function, and state the domain of the function.

a) $f(x) = \frac{1}{x}$

b) $f(x) = \frac{3x+5}{2x-6}$

c) $f(x) = \frac{2x-1}{x^2+2x-8}$

WHAT IS A HOLE?

A hole is a point of discontinuity on the graph of a function.

WHEN DOES A HOLE OCCUR?

If b is a zero of the denominator of a rational function $f(x)$ before the function $f(x)$ is reduced to its lowest terms, but b is NOT a zero of the denominator after $f(x)$ is simplified, then, a hole occurs on the graph of $f(x)$ at the point $(b, f(b))$. Note, evaluate $f(b)$ using the reduced form of $f(x)$.

Ex. Find the hole(s) of the rational function $f(x) = \frac{x-2}{x^2-4}$.

Note, vertical asymptotes and holes are both resulted by undefined domain of a rational function. This is why the graph of a rational function cannot touch or cross a vertical asymptote or a hole.

WHAT IS A HORIZONTAL ASYMPTOTE?

A horizontal asymptote (H. A.) is a horizontal line such that the graph of a function $f(x)$ approaches the horizontal line as $x \to \infty$ or $x \to -\infty$.

WHEN DOES A HORIZONTAL ASYMPTOTE OCCUR?

The H.A. of a rational function $f(x) = \frac{p(x)}{q(x)}$ occurs depending on the degrees of the function's numerator and denominator.(see the table next page)

degree of $p(x)$ and $q(x)$	**equation of the H.A.**
$deg.p(x) < deg.q(x)$	$y = 0$ (the x-axis)
$deg.p(x) = deg.q(x)$	$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$
$deg.p(x) > deg.q(x)$	no H.A

Note: A rational function has at most one H.A., and it's possible for the graph of the function to touch or cross a horizontal asymptote. However, we won't discuss this case in this class.

Ex. Find the horizontal asymptote of each rational functions.

a) $f(x) = \frac{1}{x}$

b) $f(x) = \frac{3x+5}{2x-6}$

c) $f(x) = \frac{2x^2-11}{x^2+4x-7}$

d) $f(x) = \frac{x-3}{x^2-9}$

Note: it's possible for a rational function to have an oblique asymptote, or slant asymptote. However, we will not investigate this possibility in this class.

Ex. Graph the following rational functions. Make sure to graph all asymptotes as dashed lines, label x-intercepts, y-intercepts, and some other points that can help to graph the functions.

a) $f(x) = \frac{2x+1}{x+3}$

b) $f(x) = \frac{x-2}{x^2-4}$

c) $f(x) = \frac{x-2}{x^2-x-2}$