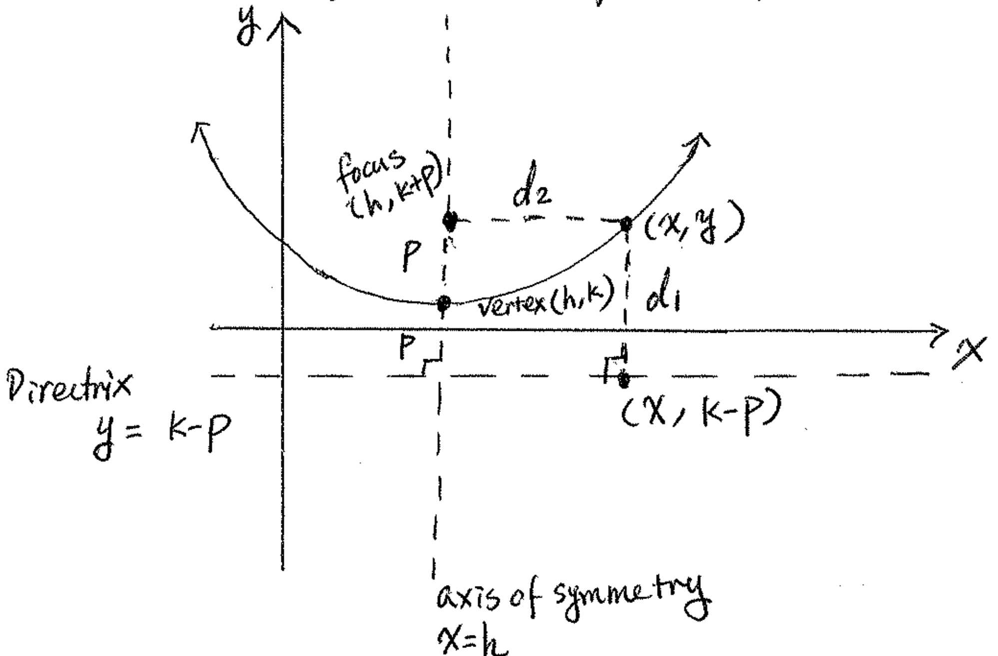
10.1 The Parabola

- Def. A parabola is the set of all points in the plane that are equidistant from a fixed line (the directrix) and a fixed point not on the line (the focus).



Note (2) The focus is "inside" the parabola, & the directrix is outside the parabola.

(1) – Both Vertex & the focus are on the axis of symetry.

(3) – The direct distance from the vertex to the focus is called focal lenth, denoted with letter "|P|".

$P > 0$, if ↻;

$P < 0$, if ↷.

Derive an equation of a parabola:

$$d_1 = d_2$$

$$\sqrt{(x-x)^2 + (y-(k-p))^2} = \sqrt{(x-h)^2 + (y-(k+p))^2}$$

$$[y-(k-p)]^2 = (x-h)^2 + [y-(k+p)]^2$$

$$\cancel{y^2} - 2y(k-p) + (k-p)^2 = (x-h)^2 + \cancel{y^2} - 2y(k+p) + (k+p)^2$$

$$2y(k+p) - 2y(k-p) = (x-h)^2 + (k+p)^2 - (k-p)^2$$

$$y(2(k+P) - 2(k-P)) = (x-h)^2 + \cancel{k^2} + 2kP + \cancel{P^2} - \cancel{k^2} + 2kP - \cancel{P^2}$$

$$y(\cancel{2k} + 2P - \cancel{2k} + 2P) = (k-h)^2 + 4kP$$

$$y = \frac{1}{4p}(k-h)^2 + k$$

Vertex Form of a quadractic eqn.

$$\boxed{y = a(x-h)^2 + k, \quad a = \frac{1}{4p}}$$

(h, k) is the vertex.

– Graphing a parabola given its focus & directrix.

Ex. Find the vertex, axis of symmetry, x-int, y-int of the parabola that has focus $(3, \frac{7}{4})$ and directrix $y = \frac{9}{4}$. sketch the graph. show focus & directrix

Solution: since directrix: $y = \frac{9}{4}$ is above the focus $(3, \frac{7}{4})$,

focus: $(h, k-p) = (3, \frac{7}{4}) \Rightarrow h = 3,$ $\qquad K = \frac{\frac{9}{4}+\frac{7}{4}}{2} = 2$

vertex: $(h, K) = \boxed{(3, 2)}$

axis of symmetry: $\boxed{x = 3}$

Equation: $y = \frac{1}{4p}(x-h)^2 + K$

$$y = \frac{1}{4(-\frac{1}{4})}(x-3)^2 + 2$$

$$y = -(x-3)^2 + 2$$

solve $y = -(x-3)^2 + 2 = 0$

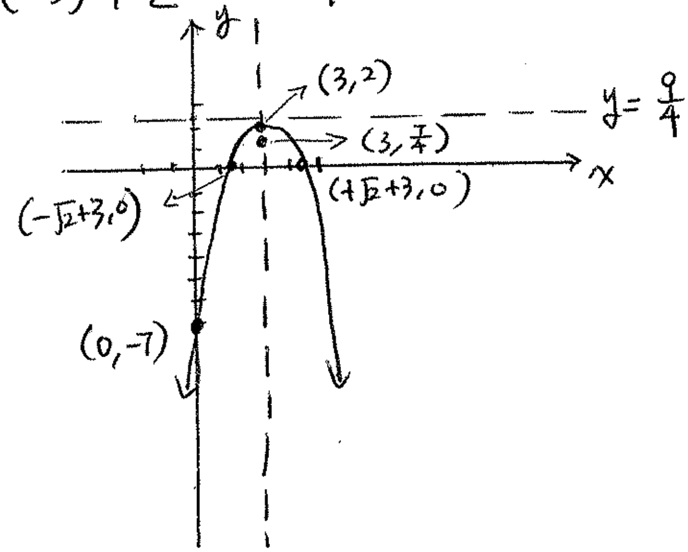
$$(x-3)^2 = 2$$

$$x = \pm\sqrt{2} + 3$$

$\boxed{\text{x-int } (\sqrt{2}+3, 0) \quad \& \quad (-\sqrt{2}+3, 0)}$

Let $x = 0$, $y = -(-3)^2 + 2 = -9 + 2 = -7$

$\boxed{\text{y-int } (0, -7)}$



Recall:

- An upward/downward parabola has an equation

$$y = ax^2 + bx + c, \quad a \neq 0,$$

which can be put in the vertex form

$$y = a(x-h)^2 + k, \quad \text{where } (h, k) \text{ is the vertex.}$$

- An parabola can also open to the left or to the right.

$$x = ay^2 + by + c, \quad a \neq 0$$

or $x = a(y-k)^2 + h$, where (h, k) is the vertex

If $a > 0$, open to the right

If $a < 0$, ... left

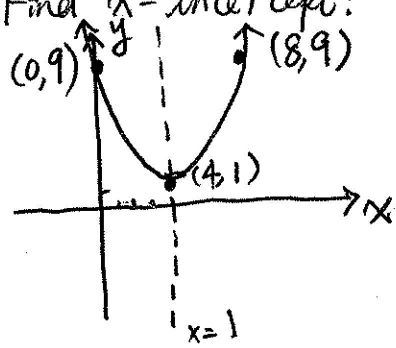
Axis of symmetry is a horizontal line passing through the vertex: $y = k$.

Ex: Find the vertex, axis of symmetry, x-intercepts, and y-intercepts.

1). $y = \frac{1}{2}(x-4)^2 + 1$

Vertex: $(4, 1)$; axis of symmetry: $x = 1$,

Find x-intercept:



$\frac{1}{2}(x-4)^2 + 1 = 0$

$(x-4)^2 = -2$

No x-intercept

Find y-int:

$y = \frac{1}{2}(-4)^2 + 1$

$= \frac{1}{2} \cdot 16 + 1$

$= 9$

$(0, 9)$.

ex. 2 $x = 3(y-1)^2 + 3$ $\quad x = a(y-k)^2 + h$

Vertex (h, k) : $(3, 1)$;

axis of symmetry: $y = 1$;

x-int : solve $x = 3(-1)^2 + 3$ $\quad (6, 0)$

Let $y=0$

$= 3 \cdot 1 + 3$

$= 6$

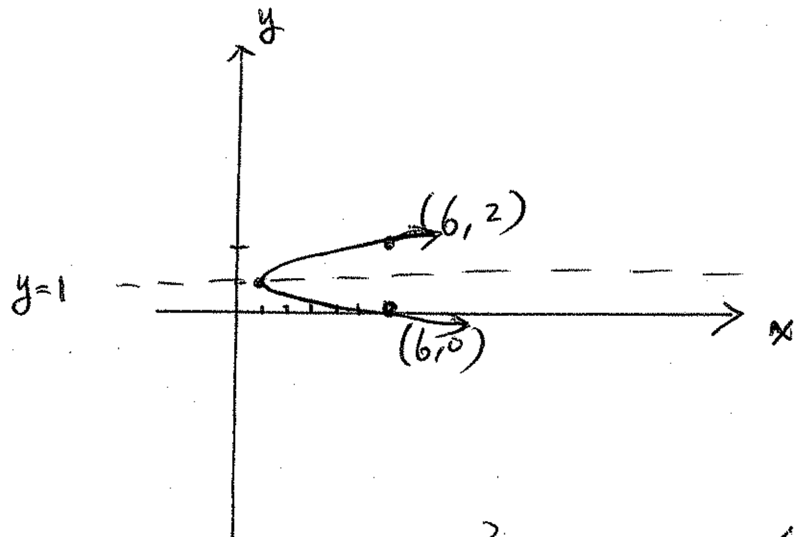
y-int : solve $3(y-1)^2 + 3 = 0$

Let $x=0$

$3(y-1)^2 = -3$

$(y-1)^2 = -1 \Rightarrow$ No real solution

(No y-int)



Ex 3: $x = -\frac{1}{2}(y+2)^2 + 1$

Vertex: $(1, -2)$; A. of symmetry: $y = -2$

~~Solve~~ Let $x=0$, $-\frac{1}{2}(y+2)^2 + 1 = 0$

$-\frac{1}{2}(y+2)^2 = -1$

$(y+2)^2 = 2$

$y + 2 = \pm\sqrt{2}$

$y = \pm\sqrt{2} - 2$

y int $(0, \sqrt{2} - 2)$

$(0, -\sqrt{2} - 2)$

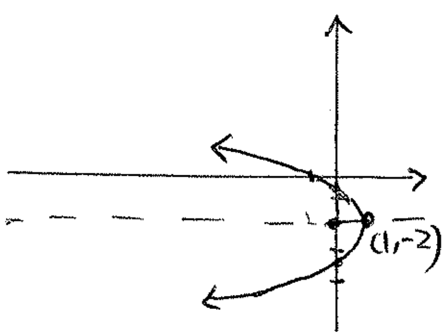
Let $y=0$ $\quad x = -\frac{1}{2}(2)^2 + 1$

$= -\frac{1}{2} \cdot 4 + 1$

$= -2 + 1$

$= -1$

x-int $(-1, 0)$



Exercise of 10.1 (Graph left & right Parabolas)

Graph the following parabolas. Label the vertex, axis of symmetry, x-intercept(s), & y-intercept(s), if exist.

1) $x=(y+3)^2$

Vertex: $\boxed{(0,-3)}$; $a=1>0 \Longrightarrow$ opens right;

axis of symmetry: $\boxed{y=-3}$;

Find x-intercept: Let $y=0 \Longrightarrow x=3^2=9$

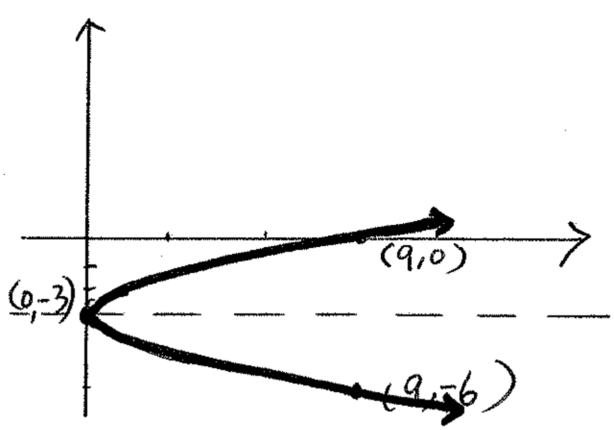
$\boxed{(9,0)}$

Find y-intercept(s): Let $x=0 \Longrightarrow (y+3)^2=0$

$y+3=0$

$y=-3$

$\boxed{0,-3}$



2) $x = -2(y-5)^2 + 2$

$a = -2 < 0 \Longrightarrow$ ⊃

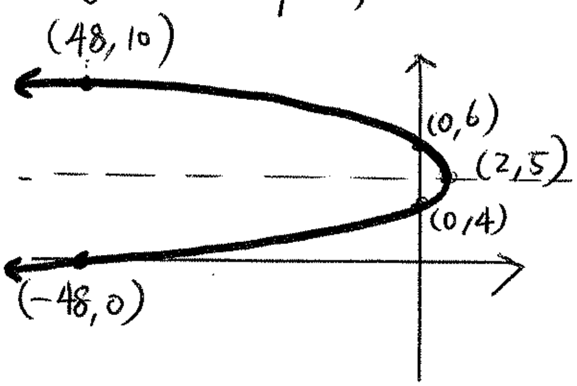
Vertex: $\boxed{(2, 5)}$; axis of symmetry: $\boxed{y = 5}$

Find x-intercept : Let $y=0$

$$x = -2(-5)^2 + 2$$
$$= -2 \cdot 25 + 2$$
$$= -48 \quad \boxed{(-48, 0)}$$

Find y-intercept(s) Let $x=0$

$$-2(y-5)^2 + 2 = 0$$
$$-2(y-5)^2 = -2$$
$$(y-5)^2 = 1$$
$$y - 5 = \pm 1$$
$$y = 6, \quad y = 4$$
$$\boxed{(0, 6) \quad (0, 4)}$$



3) $x = \frac{1}{2}(y+4)^2 + 3$

$a = \frac{1}{2} > 0 \Rightarrow$ ⊂

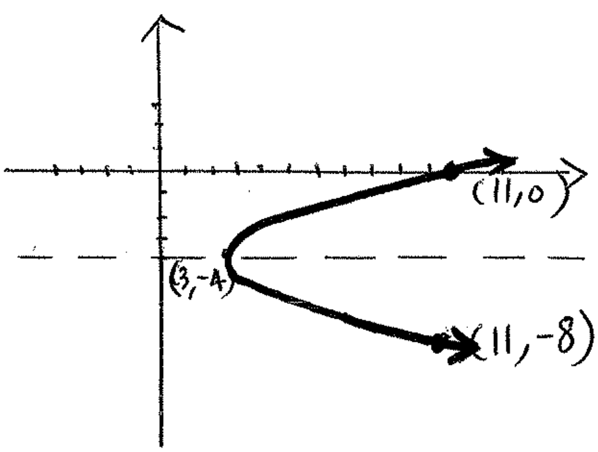
vertex: $\boxed{(3, -4)}$; axis of symmetry: $\boxed{y = -4}$

Find x-intercept : Let $y=0$, $x = \frac{1}{2} \cdot (4)^2 + 3 = \frac{1}{2} \cdot 16 + 3 = 11$ $\boxed{(11, 0)}$

Find y-intercept(s) Let $x=0$,

$$\frac{1}{2}(y+4)^2 + 3 = 0$$
$$\frac{1}{2}(y+4)^2 = -3$$
$$(y+4)^2 = -6 \quad \text{No real solution}$$

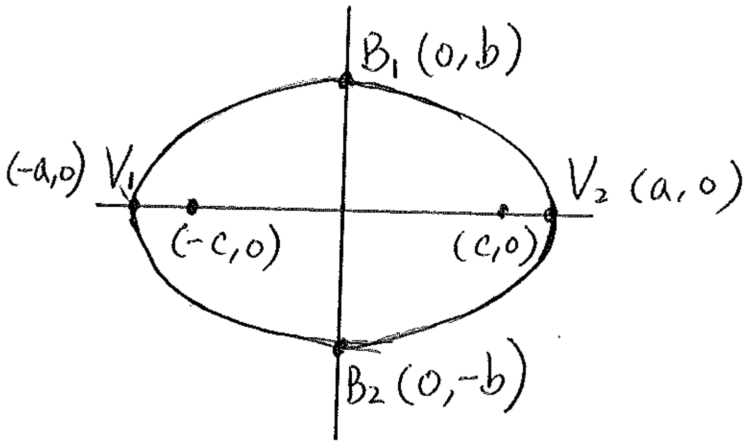
$\boxed{\text{No y-intercept.}}$



10.2 The Ellipse & the Circle

Defn: An ellipse is the set of points in a plane such that the sum of their distances from two fixed points is a constant.

(Each fixed point is called a focus, plural: foci)



The equation of an ellipse centered at the origin, with foci $(c,0)$ & $(0,-c)$, x-intercepts $(a,0)$ & $(-a,0)$, y-intercepts $(0,b)$ & $(0,-b)$ is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

where $\boxed{a^2=b^2+c^2}$

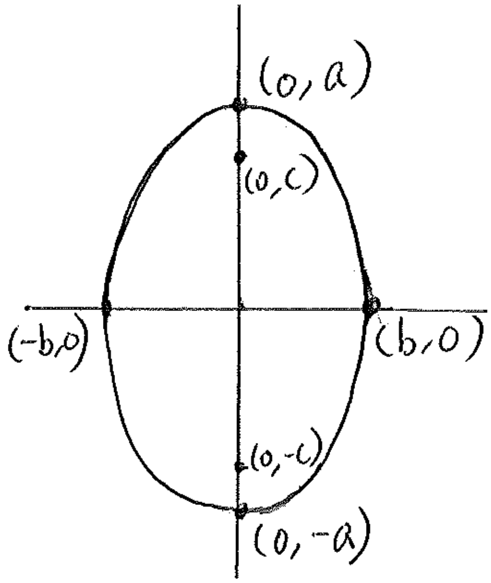
$\overline{V_1V_2}$: Major axis
$\overline{B_1B_2}$: Minor axis

} major axis is longer than minor axis

foci are on the major axis

Center: midpoint of major and minor axis

Vertices: endpoints of major axis.

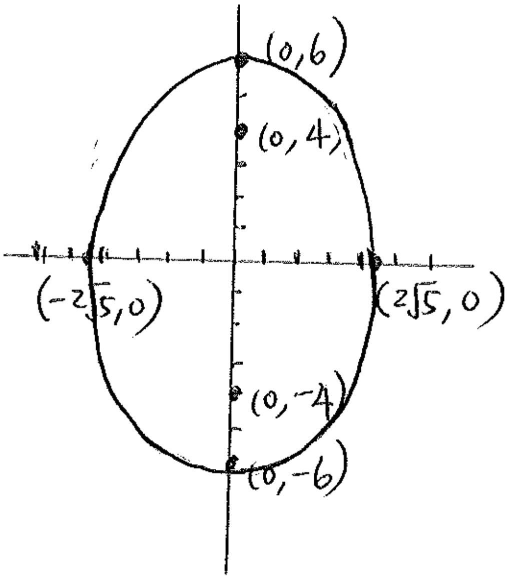


The equation of an ellipse centered at the origin with foci $(0, c)$ & $(0, -c)$, x-intercept $(b, 0)$ & $(-b, 0)$; y-intercept $(0, a)$ & $(0, -a)$ is

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

where $\boxed{a^2 = b^2 + c^2}$

Ex: sketch the ellipse with foci at $(0, 4)$ and $(0, -4)$ and Vertices $(0, 6)$ and $(0, -6)$. Find the equation of this ellipse.



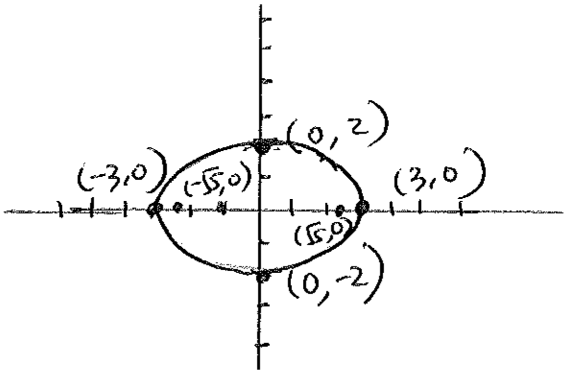
$a = 6$; $c = 4$;

$b = \sqrt{a^2 - c^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\boxed{\frac{x^2}{20} + \frac{y^2}{36} = 1}$$

Ex: sketch ~~the~~ the graph & identify ~~the~~ foci of ellipse given by: $\frac{x^2}{9} + \frac{y^2}{4} = 1$.



Let $x = 0$, $\frac{y^2}{4} = 1$ Y-intercepts:

$y^2 = 4$ $(0, 2)$ & $(0, -2)$

$y = \pm 2$

Let $y = 0$ $\frac{x^2}{9} = 1$ X-intercepts:

$x^2 = 9$ $(3, 0)$ & $(-3, 0)$.

$x = \pm 3$

$a^2 = 9$, $b^2 = 4$ $\Longrightarrow$ $c = \sqrt{a^2 - b^2}$

$= \sqrt{9-4}$

$= \sqrt{5}$

foci $(\sqrt{5}, 0)$ & $(-\sqrt{5}, 0)$

Ex: Identify ~~the~~ the type of graph given by the equation $11x^2 + 3y^2 = 66$.

$$\frac{11x^2 + 3y^2}{66} = \frac{66}{66}$$

$$\boxed{\frac{x^2}{6} + \frac{y^2}{22} = 1} \longrightarrow \text{Ellipse !}$$

Recall:

A circle is a set of all points in a plane such that their distance from a fixed point (the center) is a constant (the radius).

In fact, a circle is a special case of ellipse, where the two foci become exact one point.

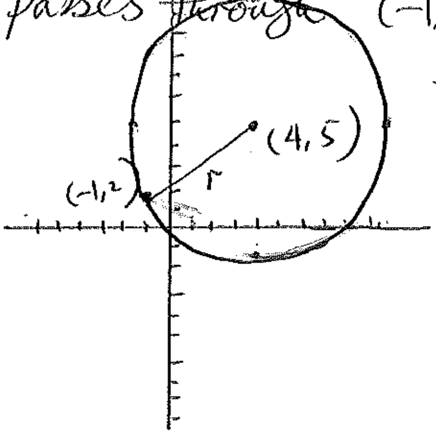
A circle centered at the origin, (0,0), with radius r is given by the equation

$$\boxed{x^2 + y^2 = r^2}$$

In general, a circle centered at any point, (h, k), with radius r is given by the equation

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

Ex: Write the equation of the circle centered at (4,5) and passes through (-1,2). Graph the circle.



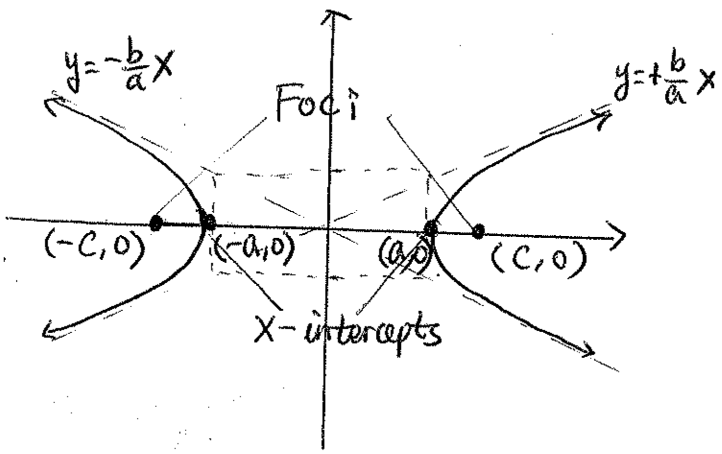
recall: distance between two points, (x_1, y_1) & (x_2, y_2): $d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

therefore: $r = \sqrt{(4-(-1))^2+(5-2)^2} = \sqrt{25+9} = \sqrt{34}$

$$\boxed{(x-4)^2+(y-5)^2 = 34}$$

10.3 The Hyperbola

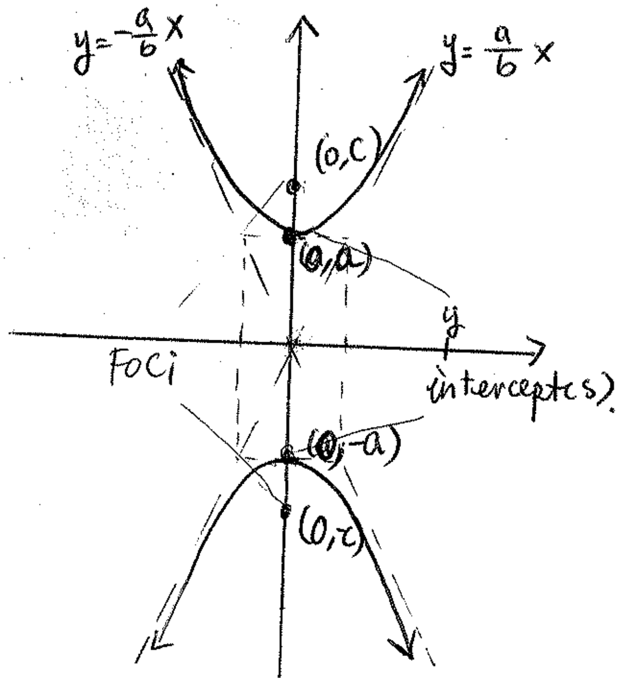
Def: A hyperbola is the set of points in a plane such that the difference between the distance from two fixed points (foci) is constant.



$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad \text{where } b^2=c^2-a^2}$$

Equation of a hyperbola centered at (0,0) opening left & right

has aymptotes: $y=\pm\frac{b}{a}x$



$$\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \quad \text{where } b^2=c^2-a^2}$$

Equation of a hyperbola centered at (0,0) opening up and down

has aymptotes $y=\pm\frac{a}{b}x$

Graphing a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (centered at (0,0); open left & right)

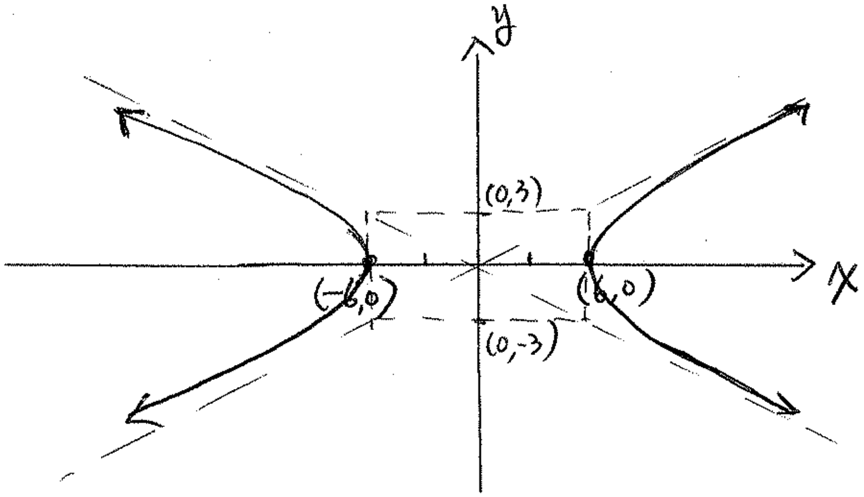
1. Locate x-intercepts $(a,0)$ & $(-a,0)$.
2. Draw a rectangle through the points $(\pm a, 0)$ and $(0, \pm b)$
(Fundamental rectangle)
3. Sketch the asymptotes by extending the diagonals of the rectangle.
4. Draw a hyperbola opening to the left & right from the x-intercepts approaching the asymptotes.

EX1: Determine the foci and the equations of the asymptotes, and sketch the graph of $\frac{x^2}{36}-\frac{y^2}{9}=1$.

$$a^2=36 \Rightarrow a=\pm 6; \qquad b^2=9 \Rightarrow b=\pm 3$$

$$c^2=a^2+b^2 \Rightarrow c=\pm\sqrt{45}$$

foci : $\boxed{(\sqrt{45},0) \text{ \& } (-\sqrt{45},0)}$



Graph the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (centered at $(0,0)$, open up and down)

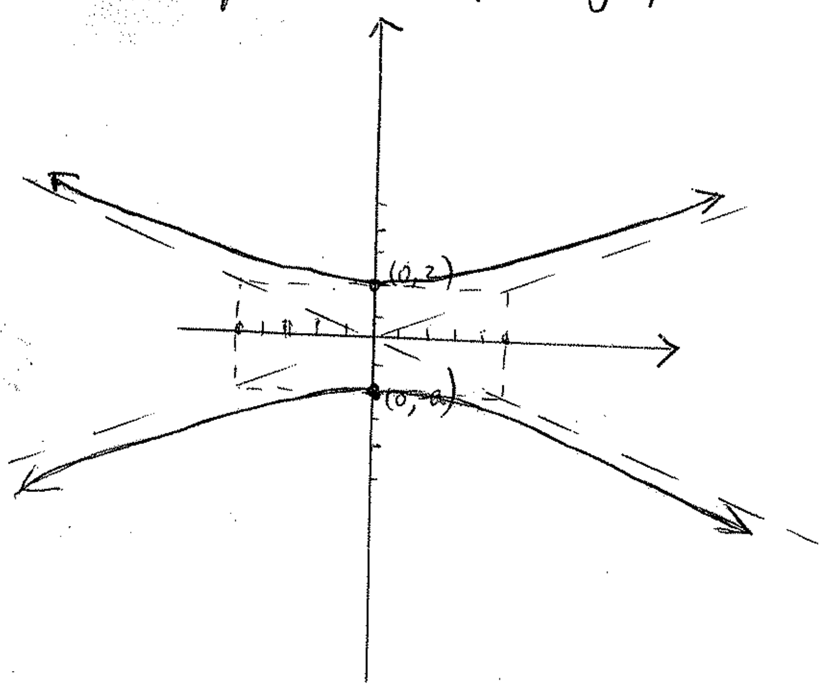
1. Locate y-intercepts $(0, a)$ and $(0, -a)$
2. Draw a rectangle ~~through~~ (fundamental rectangle) $(0, \pm a)$ and $(\pm b, 0)$
3. Sketch the asymptotes by extending ~~the~~ diagonals of the rectangle.
4. Draw a hyperbola opening up and down from ~~the~~ y-intercepts approaching the asymptotes.

ex: Determine the foci and the equations of the asymptotes. Then graph $\frac{y^2}{4} - \frac{x^2}{25} = 1$.

$C = \pm\sqrt{a^2+b^2} = \pm\sqrt{4+25} = \pm\sqrt{29}$ foci: $\boxed{\begin{matrix}(0, \sqrt{29}) \\ (0, -\sqrt{29})\end{matrix}}$

equations of asymptotes: $y = \pm \frac{a}{b} x$

$\boxed{y = \pm \frac{2}{5} x}$



Ex 3: Find the equation of the hyperbola with asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and x-intercepts $(6, 0)$ and $(-6, 0)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Asymptotes: $y = \pm \frac{b}{a}x = \pm \frac{b}{6}x = \pm \frac{1}{2}x \Rightarrow b =$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Longrightarrow \boxed{\frac{x^2}{36} - \frac{y^2}{9} = 1}$$

Ex 4: Find the equation of the hyperbola whose vertices of the fundamental rectangle are $(1, \pm 7)$ and $(-1, \pm 7)$ and opening up and down.

$a = 7, \Rightarrow a^2 = 49; \quad b = 1 \Longrightarrow b^2 = 1$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Longrightarrow \frac{y^2}{49} - \frac{x^2}{1} = 1$$

$$\boxed{\frac{y^2}{49} - x^2 = 1}$$

Exercise of 10.1 (Graph left & right Parabolas)

Graph the following parabolas. Label the vertex, axis of symmetry, x-intercept(s), & y-intercept(s), if exist.

1) $x=(y+3)^2$

Vertex: $\boxed{(0,-3)}$; $a=1>0 \Longrightarrow$ opens right;

axis of symmetry: $\boxed{y=-3}$;

Find x-intercept: Let $y=0 \Longrightarrow x=3^2=9$

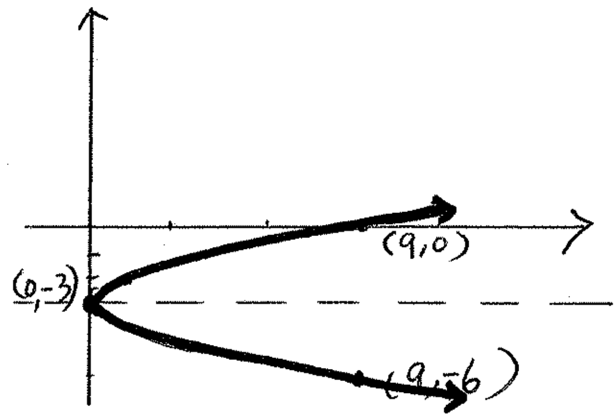
$\boxed{(9,0)}$

Find y-intercept(s): Let $x=0 \Longrightarrow (y+3)^2=0$

$y+3=0$

$y=-3$

$\boxed{0,-3}$



2) $x = -2(y-5)^2 + 2$

$a = -2 < 0 \Longrightarrow$

Vertex: $\boxed{(2, 5)}$; axis of symmetry: $\boxed{y=5}$

Find x-intercept : Let $y=0$

$x = -2(-5)^2 + 2$

$= -2 \cdot 25 + 2$

$= -48$ $\boxed{(-48, 0)}$

Find y-intercept(s) Let $x=0$

$-2(y-5)^2 + 2 = 0$

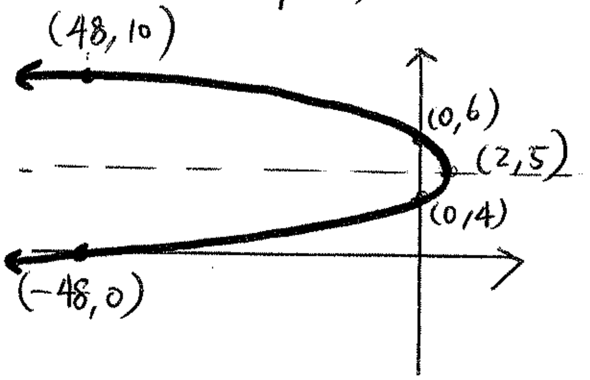
$-2(y-5)^2 = -2$

$(y-5)^2 = 1$

$y - 5 = \pm 1$

$y = 6, \quad y = 4$

$\boxed{(0,6) \quad (0,4)}$



3) $x = \frac{1}{2}(y+4)^2 + 3$

$a = \frac{1}{2} > 0 \Longrightarrow$

vertex: $\boxed{(3,-4)}$; axis of symmetry: $\boxed{y=-4}$

Find x-intercept: Let $y=0$, $x = \frac{1}{2}\cdot(4)^2 + 3 = \frac{1}{2}\cdot 16 + 3 = 11$ $\boxed{(11,0)}$

Find y-intercept(s) Let $x=0$,

$\frac{1}{2}(y+4)^2 + 3 = 0$

$\frac{1}{2}(y+4)^2 = -3$

$(y+4)^2 = -6$ No real solut

$\boxed{\text{No y-intercept}}$