1.1 Equations in One Variable

- Equation is a statement that two expressions are equal.
- Solve an equation means to solve the unknowns that makes the equation true.

In this section, we'll study solving linear equations, rational equations and absolute value equations.

Solve linear equations

Definition

A linear equation in one variable has the form: $ax + b = 0$, where a and b are real numbers, and $a \neq 0$.

Principles of solving linear equations:

If A, B and C are algebraic expressions and C is a real number, and $A = B$, then following are true:

- 1) $A + C = B + C$
- 2)$A - C = B - C$
- 3) $AC = BC$
- 4) $\frac{A}{C} = \frac{B}{C}$, for $C \neq 0$

Ex. Solve the following linear equations.

- a) $3(4x - 1) = 4 - 6(x - 3)$
- b) $\frac{1}{2}x - 6 = \frac{3}{4}x - 9$

Rational expressions and equations

Definition

A rational equation contains rational expressions. i.e.

$$\frac{x-8}{3}+\frac{x-3}{2}=0, \quad 3-\frac{a-2}{b}=7.$$

Technique of solving rational equations:

- Multiplying both sides of the equation by the least common denominator (LCD) to clear away fractions.

Ex. Solve the following rational equations, and check your solution(s).

- a) $\frac{x+4}{x+1} = \frac{x+4}{x+5}$
- b) $\frac{y}{y-3} + 3 = \frac{3}{y-3}$
- c) $\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$
- d) $\frac{1}{2} + \frac{1}{x-1} = 1$

Warning: Since rational equations may contain variables in the denominators, you always need to make sure that the denominators CANNOT be 0!

Solve Equations with Absolute Value

Definition

The absolute value of a number or an algebraic expression A, denoted as $|A|$, is its distance from 0 on the number line.

$$
\begin{aligned}
|A| = k \quad \text{for } k > 0 \quad &\Longrightarrow \quad A = -k \text{ or } A = k \\
|A| = 0 \quad &\Longrightarrow \quad A = 0 \\
|A| = k \quad \text{for } k < 0 \quad &\Longrightarrow \quad \text{No solution}
\end{aligned}
$$

Ex. Solve the following absolute value equations.

- a) $|x| = 5$.
- b) $|x - 5| = 4$.
- c) $2|x + 8| - 6 = 0$.

Techniques for solving absolute value equations:

- First isolate the absolute value on one side of the equation.
- Then use definition of absolute value to solve the equation.

Section 1.3: Distance, Mid-point, and Circle

Distance formula and Midpoint

- Definition

The distance d between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Definition

The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) has the coordinate

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - Ex. Find the distance between the two points $(-4, -7)$ and $(4, 8)$, then find the midpoint of the line segment joining the two points.

The circle

- Every point on a circle is equal-distant to its center. The fixed distance from a point on the circle to its center is called the radius.
- The equation for a circle with center (h, k) and radius r for $r > 0$ is

$$(x - h)^2 + (y - k)^2 = r^2$$

- Thus, the equation of a circle centered at the origin $(0, 0)$ is

$$x^2 + y^2 = r^2$$

- Ex 1. Determine the center and radius of $x^2 + (y - 2)^2 = 16$.
- Ex 2. Sketch the graph of the equation $(x - 1)^2 + (y + 2)^2 = 9$.
- Ex 3. Write the standard equation for the circle centered at $(-2, 5)$ with radius $\frac{1}{2}$.

Section 1.4 Linear Equations in Two Variables

- The slope of a line, denoted as m, measures the steepness of the line.
- The slope m of a line containing points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{y_1 - y_2}{x_1 - x_2} \end{aligned}$$

- Ex.1. Find the slope of the line that passes through the points $(-3, 7)$ and $(5, -1)$.

Classification of Lines by Slope

Line	Slope of the line
Increasing from left to right /	$m > 0$
Decreasing from left to right \	$m < 0$
Horizontal —	$m = 0$
Vertical |	Undefined

How to write an equation of a line?

The equation of a straight line is called a linear equation. A linear equation can be written in three forms.

- point-slope form: $y - y_1 = m(x - x_1)$

  where m is the slope of the line and (x_1, y_1) is a point on the line.

- slope-intercept form: $y = mx + b$

  where m is the slope and $(0, b)$ is the y-intercept.

- standard form: $Ax + By = C$

  where A, B and C are integers.

- Ex.1. Using the point-slope form, write an equation of the line containing the two points $(-10, 4)$ and $(-2, 0)$. Then rewrite the equation to its slope-intercept form and standard form.

- Ex.2. Identify the slope and y-intercept from the linear equations.

  a) $y = -x - 5$ b) $y = 2x$.

- Ex.3. Find the slope and y-intercept for $3x + 5y = 15$.

Graph of a linear equation

You can graph a straight line, if

- 1) you have the coordinates of two points on that line.
- 2) Or, if you have the coordinate of one point on that line, and the slope of the line.

Ex. Use both methods to graph the following linear equations.

- a) $x + 2y = -6$
- b) $y = -\frac{3}{5}x$

Horizontal and Vertical Line

- A horizontal line is given by the equation $y = b$, where the slope is 0, and $(0, b)$ is the y-intercept.
- A vertical line is given by the equation $x = a$, its slope is undefined, and $(a, 0)$ is the x-intercept.

Ex. Sketch the line $y = -6$ and the line $y = 0$, $x = 2$, $x = 0$ on the same graph.

Slope of parallel lines and perpendicular lines

- If two lines l_1, l_2, are parallel, they must have the same slope.

$$m_1 = m_2$$

- If two lines l_1, l_2, are perpendicular, their slopes are negative reciprocal of each other.

$$m_1 = -\frac{1}{m_2}$$

Note,

- Vertical lines are parallel, even though their slopes are undefined.
- A vertical line and a horizontal line are perpendicular.

- Ex. a) Write an slope-intercept equation of the line containing the point $\left(\frac{1}{3}, \frac{3}{4}\right)$, and is parallel to the graph of another line $y = \frac{2}{3}x + 5$.

  b)Write an slope-intercept equation of the line containing the same point as in part a), but is perpendicular to the graph of the line $y = \frac{2}{3}x + 5$.

1.7 Linear and Absolute Value Inequalities

- An inequality contains $<, >, \leq,$ or $\geq$. For example, $3x - 5 < 6 - 2x$ is an inequality.
- To solve an inequality is to find all values of the variable that make the inequality true. Each of these numbers is a solution of the inequality, and the set of all such solutions is its solution set.
- Two inequalities have the same solution set are called equivalent inequalities.

Properties of inequality

If A, B and C are algebraic expressions and C is a real number, and $A < B$, then following are true:

- 1) $A \pm C < B \pm C$
- 2) $AC < BC$, if $C > 0$; and $AC > BC$, if $C < 0$;
- 3) $\frac{A}{C} < \frac{B}{C}$, for $C > 0$; and $\frac{A}{C} > \frac{B}{C}$, for $C < 0$

Similar statement hold for $A \leq B$.

In summary, when **multiplying or dividing a negative number** on both side of an inequality, you must **reverse the inequality symbol**.

- For example: $-2 < 3$, and $(-2)(-4) > (3)(-4)$ because $8 > -12$;

  Another example, $10 > 5$, and $\frac{10}{-5} < \frac{5}{-5}$ because $-2 < -1$.

Ex.1. Solve each of the following, graph the solution set and leave the solution(s) in set notation and in interval notation.

- a) $3x - 5 < 6 - 2x$
- b) $\frac{1}{4}x - 3 \geq \frac{1}{2}x + 2$

Note:

1) $<$ and $>$ $\Longrightarrow$ open interval $\Longrightarrow$ $\circ$ on the graph $\Longrightarrow$ (or) in the interval notation.

2) $\leq$ and $\geq$ $\Longrightarrow$ closed interval $\Longrightarrow$ $\bullet$ on the graph $\Longrightarrow$ [or] in the interval notation.

3)Use a parenthesis for ∞ or $-\infty$, because ∞ is always open.

Compound Inequalities

A compound inequality is formed by joining two inequalities with the word *and* or the word *or*.

For example: The two joined inequalities

- $-3 < 2x + 5$ *and* $2x + 5 \leq 7$ is called a conjunction, which can be abbreviated as $-3 < 2x + 5 \leq 7$.
- $2x + 5 > -3$ *or* $2x + 5 < -10$ is called a disjunction.

Inequalities with Absolute Value

Definition

For $k > 0$ and an algebraic expression X:

- $|X| < k$ means the distance from X to 0 is less than k units, thus

$$|X| < k \text{ is equivalent to } -k < X < k.$$

- $|X| > k$ means the distance from X to 0 is greater than k units, thus

$$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k.$$

Similar statements hold for $|X| \leq k$ and $|X| \geq k$.

For example:

- $|x| < 3$ is equivalent to $-3 < x < 3$;

  $|2y + 3| \geq 4$ is equivalent to $2y + 3 \leq -4$ or $2y + 3 \geq 4$.

Ex.1. Solve each of the following, graph the solution set and leave the solution(s) in a set notation or in an interval notation.

a) $|3x + 2| < 7$

b) $-2|4 - x| \leq -4$

c) $|7x - 9| + 3 \geq 0$

d) $|\frac{x-6}{5}| > 1$

Techniques to solve absolute value inequalities:

1) Always start by isolating the absolute value on one side of the inequality.

2) Use definition to solve the inequality.