Review of methods for finding the particular solution of a nonhomogeneous system

Everything in this review is based on the following:

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$$

This is also written

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\mathbf{x}'} = \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}}_{\mathbf{P}(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}}_{\mathbf{g}(t)}$$

When discussing the general solution to this system, I will use the following notation:

- Eigenvalues: r_1, r_2, etc.
- Eigenvectors:

$$\xi^1 = \begin{bmatrix} \xi_1^1(t) \\ \xi_2^1(t) \end{bmatrix} \text{ for } r_1, \text{ and } \xi^2 = \begin{bmatrix} \xi_1^2(t) \\ \xi_2^2(t) \end{bmatrix} \text{ for } r_2.$$

- General solution:

$$\begin{aligned} \mathbf{x}_g(t) &= c_1\xi^1 e^{r_1 t} + c_2\xi^2 e^{r_2 t} \\ &= c_1 \begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} \xi_{12} \\ \xi_{22} \end{bmatrix} e^{r_2 t} \end{aligned}$$

  Recall: $\xi_1^2 = \xi_{12}, \xi_2^1 = \xi_{21}$, etc.
- Particular solution: $\mathbf{x}_p(t)$.

I will try to keep the notation consistent throughout, although I may not include the (t) every time.

I am assuming that you have already found the general solution of the system, in everything that follows.

1. Diagonalization

Summary: The principle of diagonalization is to exploit the magic of linear algebra to make your computations simpler. Specifically, *if* $\mathbf{P}$ *is diagonalizable,* let $\mathbf{x} = \mathbf{T}\mathbf{y}$ so that you can solve the simpler system

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}. \tag{1}$$

Advantages:

a. This system is simpler because $\mathbf{D}$ is a diagonal matrix.

b. $\mathbf{D}$ does not need to be calculated. Although you can find it from computing $\mathbf{T^{-1}PT}$, this computation is unnecessary. $\mathbf{D}$ is just the matrix of eigenvalues, which you have already found:

$$\mathbf{D} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

c. $\mathbf{T}$ has also already been found: it is just the matrix of eigenvalues. Note: $\mathbf{T}$ is not *quite* the same as the fundamental matrix $\mathbf{\Psi}$ because *it does not include the functions* $e^{r_1 t}, e^{r_2 t}$*, etc.*

Disadvantages:

a. The matrix $\mathbf{P}$ must be diagonalizable. This means that it must have n linearly independent eigenvectors if $\mathbf{P}$ is $n \times n$. Remember: $\mathbf{P}$ is automatically diagonalizable if it is Hermitian, that is:

$$\mathbf{P} = \overline{\mathbf{P}}^T.$$

If $\mathbf{P}$ contains only real numbers, then this just means that $\mathbf{P}$ is automatically diagonalizable whenever $\mathbf{P}$ is symmetric:

$$\mathbf{P} = \mathbf{P}^T.$$

How to use the method of diagonalization:

1. Write down the matrix of eigenvectors $\mathbf{T}$ and find its inverse:

$$\mathbf{T} = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \qquad \mathbf{T}^{-1} = \tfrac{1}{\xi_{11}\cdot\xi_{22} - \xi_{12}\cdot\xi_{21}} \begin{bmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{bmatrix}.$$

2. Compute

$$\mathbf{T}^{-1}\mathbf{g} = \tfrac{1}{\xi_{11}\cdot\xi_{22} - \xi_{12}\cdot\xi_{21}} \begin{bmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.$$

3. Optional step: use the principle of diagonalization (1). Write out $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} r_1 y_1 + h_1(t) \\ r_2 y_2 + h_2(t) \end{bmatrix}$$

This step is not strictly necessary, but sometimes it's good to keep the formalism in mind as a way of remembering where you are and what you're doing.
Also, it shows the TA that you know what's up.

4. Find y_1, y_2 by evaluating the indefinite integrals:

$$y_1 = e^{r_1 t} \int e^{-r_1 s} h_1(s)\, ds$$

$$y_2 = e^{r_2 t} \int e^{-r_2 s} h_2(s)\, ds$$

   Tip: don't need to bother keeping track of the constants of integration. They disappear at the end anyway and they just make more work in the mean time.

5. Now that you have $\mathbf{y}$, you can find the particular solution to the original problem by computing

$$\mathbf{x}_p = \mathbf{T}\mathbf{y} = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Common mistakes to avoid:

1. In step 4, integrate only the $e^{-r_1 s} h_j(t)$. Don't include the $r_j y_j$ part.
2. In step 4, don't forget to multiply the integral by $e^{r_1 t}$.
3. **DON'T forget step 5.** For some reason, people really like to forget step 5.

2. Undetermined Coefficients

Summary: The principle of undetermined coefficients is to exploit the tractability of the functions in $\mathbf{g}(t)$ to make your computations simpler. Specifically, we assume that a particular solution must look like

$$\mathbf{x}_p = \mathbf{a}\gamma_1(t) + \mathbf{b}\gamma_2(t) + \mathbf{d}\gamma_3(t) + \mathbf{e}. \tag{2}$$

By comparing the coefficients of the $\gamma_1(t)$ term we can figure out what $\mathbf{a}$ is, etc.

Advantages:

a. Easy mathematics: involves NO integration.

b. Can be used to find a particular solution without having to find eigenvectors.

Disadvantages:

a. Only works when $\mathbf{g}$ contains nice functions (polynomials, exponentials, trigs).

b. $\mathbf{P}$ must have constant coefficients.

How to use the method of undetermined coefficients:

1. Write out

$$\mathbf{g}(t) = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}\gamma_1(t) + \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}\gamma_2(t) + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}\gamma_3(t).$$

Here, α_j, β_j are numbers and γ_j are functions of t like $e^{kt}, \cos kt, \sin kt, t, t^2$, etc. I've written $\mathbf{g}$ as if it had three different terms, but of course it probably only has one or two different terms (and could theoretically have many more).

2. Use the principle of undetermined coefficients (2) and write out

$$\mathbf{x}_p = \mathbf{a}\gamma_1(t) + \mathbf{b}\gamma_2(t) + \mathbf{d}\gamma_3(t) + \mathbf{e}.$$

Note: if an eigenvalue $r = r_i$ shows up in a $\gamma_j(t) = e^{rt}$, you must assume $\mathbf{x}$ includes an extra term with a t, e.g.,

$$\mathbf{x}_p = \mathbf{a}e^{rt} + \mathbf{b}te^{rt} + \mathbf{d}\gamma_2(t) + \mathbf{e}\gamma_3(t) + \mathbf{f}.$$

3. Find the derivative

$$\mathbf{x}_p' = \mathbf{a}\gamma_1'(t) + \mathbf{b}\gamma_2'(t) + \mathbf{d}\gamma_3'(t).$$

4. Rewrite $\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$ as

$$\underbrace{\mathbf{a}\gamma_1' + \mathbf{b}\gamma_2' + \mathbf{d}\gamma_3'}_{\mathbf{x}_p'} = \underbrace{\mathbf{P}\mathbf{a}\gamma_1 + \mathbf{P}\mathbf{b}\gamma_2 + \mathbf{P}\mathbf{d}\gamma_3 + \mathbf{P}\mathbf{e}}_{\mathbf{P}\mathbf{x_p}} + \underbrace{\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}\gamma_1 + \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}\gamma_2 + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}\gamma_3}_{\mathbf{g}}.$$

5. Collect all the γ_j terms to obtain equations like

$$\mathbf{P}\mathbf{a}\gamma_1 = \mathbf{a}\gamma_1' - \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}\gamma_1 \quad \Longrightarrow \quad \mathbf{P}\mathbf{a} = \mathbf{a} - \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}.$$

Note: these equations may look quite different from what I've written here, e.g., $\mathbf{P}\mathbf{a}$ may not always be with $\mathbf{a}\gamma_1'$. The important thing is to gather all the terms with γ_1 together, all the terms with γ_2 together, etc. Then you can make an equation between the coefficients of the γ_j. (This is where the name of this method comes from, right?)

6. Solve these equations to find a_1, a_2:
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p_{11}a_1 + p_{12}a_2 \\ p_{21}a_1 + p_{22}a_2 \end{bmatrix} = \begin{bmatrix} a_1 - \alpha_1 \\ a_2 - \beta_1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
Sometimes, you may not be able to solve for **a** completely. For example, suppose you obtain
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_1 - \frac{1}{5} \end{bmatrix}$$
from the equations you obtained in part 5. In this case, follow the book and write
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_1 - \frac{1}{5} \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{5} \end{bmatrix}.$$
Then set $a_1 = 0$ and take
$$\mathbf{a} = \begin{bmatrix} 0 \\ -\frac{1}{5} \end{bmatrix}.$$
Also note that sometimes you may need to solve for **b** before you can solve for **a**, etc.

7. Now that you have found $\mathbf{a}, \mathbf{b}$,etc., write out the particular solution
$$\mathbf{x}_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \gamma_1(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \gamma_2(t) + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \gamma_3(t) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$
Note: this is the particular solution, not the general solution. THERE ARE NO CONSTANTS c_1, c_2 in front of these terms.

Common mistakes to avoid:

1. **DON'T forget the g in step 4.** People love to forget the **g** in step 4.
2. The **g** starts off on the same side as the **Px** is in step 4. It needs to be **subtracted** over to the side in step 5. Many people make a sign error at this point.

3. Variation of Parameters

Summary: exploit the properties of the fundamental matrix $\mathbf{\Psi}$ by solving the equation

$$\mathbf{\Psi}\mathbf{u}' = \mathbf{g}. \tag{3}$$

to obtain a vector $\mathbf{u}$. Then the particular solution is $\mathbf{x}_p = \mathbf{\Psi}\mathbf{u}$.

Advantages:

a. May be applied to solve any problem.

b. Easy to remember: the only formulae you need are are $\mathbf{\Psi}\mathbf{u}' = \mathbf{g}$ and $\mathbf{x}_p = \mathbf{\Psi}\mathbf{u}$.

Disadvantages:

a. Solving $\mathbf{\Psi}\mathbf{u}' = \mathbf{g}$ can get ugly because each component of the vectors involved can contain a complicated expression and row-reduction becomes cumbersome.

b. Integrating $\mathbf{u}'$ to find $\mathbf{u}$ can sometimes be difficult.

c. Doing the matrix multiplication $\mathbf{\Psi}\mathbf{u}$ can be tedious.

How to use the method of variation of parameters:

1. Use your general solution to form the fundamental matrix, which probably looks like this:
$$\mathbf{\Psi} = \begin{bmatrix} \xi_{11}e^{r_1 t} & \xi_{12}e^{r_2 t} \\ \xi_{21}e^{r_1 t} & \xi_{22}e^{r_2 t} \end{bmatrix}.$$
2. Use the principle of variation of parameters (3) and solve the matrix equation $\mathbf{\Psi}\mathbf{u}' = \mathbf{g}$ to find $\mathbf{u}'$:
$$\begin{bmatrix} \xi_{11}e^{r_1 t} & \xi_{12}e^{r_2 t} \\ \xi_{21}e^{r_1 t} & \xi_{22}e^{r_2 t} \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}.$$
3. Integrate u_1' to find u_1, and integrate u_2' to find u_2.
Tip: You don't need to worry about the constants of integration here, as they will magically disappear in the last step anyway.
4. Now that you have $\mathbf{u}$, perform the matrix multiplication $\mathbf{\Psi}\mathbf{u}$ to recover $\mathbf{x}_p$:
$$\begin{bmatrix} \xi_{11}e^{r_1 t} & \xi_{12}e^{r_2 t} \\ \xi_{21}e^{r_1 t} & \xi_{22}e^{r_2 t} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{x}_p(t).$$

Common mistakes to avoid:

1. **Don't forget step 4.** People love to forget step 4.