THE SIMPLEX ALGORITHM

1. How to Apply the Simplex Algorithm

Solve via the simplex algorithm:

$$\begin{array}{rrrrl} \text{maximize} & x_1 & +2x_2 & & \\ \text{subject to} & 2x_1 & -x_2 & \leq & 12 \\ & -x_1 & +x_2 & \leq & 3 \\ & 2x_1 & +3x_2 & \leq & 24 \\ & & x_i & \geq & 0. \end{array}$$

First, we put this into standard/computational form:

$$\begin{array}{rrrrrrrl} \text{minimize} & -x_1 & -2x_2 & & & & & \\ \text{subject to} & 2x_1 & -x_2 & +x_3 & & & = & 12 \\ & -x_1 & +x_2 & & +x_4 & & = & 3 \\ & 2x_1 & +3x_2 & & & +x_5 & = & 24 \\ & & & & & x_i & \geq & 0. \end{array}$$

That is, minimize $f(\boldsymbol{x}) = \boldsymbol{b}\boldsymbol{x}$ subject to $A\boldsymbol{x} = \boldsymbol{c}$, where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 12 \\ 3 \\ 24 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix}.$$

Create the simplex tableau:

b_1	b_2	b_3	b_4	b_5	
-1	-2	0	0	0	
a_1	a_2	a_3	a_4	a_5	c
2	-1	1	0	0	12
-1	1	0	1	0	3
2	3	0	0	1	24

The coefficients of the objective function are written across the top. These b_i are the only thing that never changes while the algorithm runs. Below it is A, with $\boldsymbol{c}$ adjoined at the far right.

(1) Find the feasible basis. Start with a_3, a_4, a_5 as a basis, because it consists of basis vectors. Write the basis vectors at left. Even further to the left, copy down the coefficients b_i corresponding to the current basis vectors a_i.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
φ		a_1	a_2	a_3	a_4	a_5	c
0	a_3	2	-1	1	0	0	12
0	a_4	-1	1	0	1	0	3
0	a_5	2	3	0	0	1	24

(2) Compute the gutter, using $\varphi \cdot a_i - b_i$.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
φ		a_1	a_2	a_3	a_4	a_5	c
0	a_3	2	-1	1	0	0	12
0	a_4	-1	1	0	1	0	3
0	a_5	2	3	0	0	1	24
		1	2	0	0	0	0

I.e., the bottoms entries are

$$\begin{aligned}
1 &= 0(2) + 0(-1) + 0(2) - (-1) \\
2 &= 0(-1) + 0(1) + 0(3) - (-2) \\
0 &= 0(1) + 0(0) + 0(0) - 0 \\
0 &= 0(0) + 0(1) + 0(0) - 0 \\
0 &= 0(0) + 0(0) + 0(1) - 0 \\
0 &= 0(12) + 0(3) + 0(24)
\end{aligned}$$

This final 0 at the far right indicates the current value of the function, i.e., when $x_1, x_2 = 0$.

At left, only a_3, a_4, a_5 appear, i.e., a_1, a_2 don't have coefficients appearing in the outside box at far right. This means $x = [0, 0, 12, 3, 24]$ is a feasible solution (with value 0). Projecting back to $\mathbb{R}^2$, we are at the extreme point $x = (0, 0)$ of the feasible set; see Figure 1, below.

(3) Find the largest positive entry in the gutter. (If nothing is positive, algorithm stops.) In this case, it is the circled 2. This tells you which column a_i contains your pivot (a_2 in this example).

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
		a_1	a_2	a_3	a_4	a_5	c
0	a_3	2	-1	1	0	0	12
0	a_4	-1	1	0	1	0	3
0	a_5	2	3	0	0	1	24
		1	(2)	0	0	0	0
			⇑				

(4) Find the pivot by selecting the smallest positive ratio c_j/a_{ij}. In this case, it is the circled 3. Since it occurs in the second row, your pivot is the circled 1.

		b_1	b_2	b_3	b_4	b_5		
		-1	-2	0	0	0		
		a_1	a_2	a_3	a_4	a_5	c	
0	a_3	2	-1	1	0	0	12	$c_1/a_{12} = 12/-1 = -12 \leq 0$•
0	a_4	-1	(1)	0	1	0	3	$c_2/a_{22} = 3/1 =$ (3) ⇐
0	a_5	2	3	0	0	1	24	$c_3/a_{32} = 24/3 = 8$
		1	2	0	0	0	0	

(5) Use row reduction to change a_2 into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. In this case, the pivot is already 1.

b_1	b_2	b_3	b_4	b_5	
-1	-2	0	0	0	
a_1	a_2	a_3	a_4	a_5	c
1	0	1	1	0	15
-1	1	0	1	0	3
5	0	0	-3	1	15

We have completed one full iteration of the simplex algorithm. Now start over.

(1) Find the feasible basis, i.e., label the left edge with a_3, a_2, a_4, since a_2 is now in the basis. a_2 is now in the basis, so put $b_2 = -2$ (the coefficient directly above a_2) at far left, next to a_2. The other two are still the same.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
		a_1	a_2	a_3	a_4	a_5	c
0	a_3	1	0	1	1	0	15
-2	a_2	-1	1	0	1	0	3
0	a_5	5	0	0	-3	1	15

(2) Compute the gutter, using $\varphi \cdot a_i - b_i$.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
φ		a_1	a_2	a_3	a_4	a_5	c
0	a_3	1	0	1	1	0	15
-2	a_2	-1	1	0	1	0	3
0	a_5	5	0	0	-3	1	15
		3	0	0	-2	0	-6

This tableau means $x = [0, 3, 15, 0, 15]$ is a feasible solution (with value -6). The coordinates of the basis vectors a_2, a_3, a_5 which appear in the solution are the corresponding numbers under c. Projecting back to $\mathbb{R}^2$, we are at the extreme point $x = (0, 3)$ of the feasible set; see Figure 1.

(3) Find the largest positive entry in the gutter. (If nothing is positive, algorithm stops.) In this case, it is the 3. So column a_1 contains your pivot.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
		a_1	a_2	a_3	a_4	a_5	c
0	a_3	1	0	1	1	0	15
-2	a_2	-1	1	0	1	0	3
0	a_5	5	0	0	-3	1	15
		③	0	0	-2	0	-6
		⇑					

(4) Find the pivot by selecting the smallest positive ratio c_j / a_{ij}. In this case, it is the circled 3. Since it occurs in the third row, your pivot is the circled 5.

		b_1	b_2	b_3	b_4	b_5		
		-1	-2	0	0	0		
		a_1	a_2	a_3	a_4	a_5	c	
0	a_3	1	0	1	1	0	15	15
-2	a_2	-1	1	0	1	0	3	•(negative)
0	a_5	⑤	0	0	-3	1	15	③⇐
		3	0	0	-2	0	-6	

(5) Use row reduction to change a_1 into $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

b_1	b_2	b_3	b_4	b_5	
-1	-2	0	0	0	
a_1	a_2	a_3	a_4	a_5	c
0	0	1	$8/5$	$-1/5$	12
0	1	0	$2/5$	$1/5$	6
1	0	0	$-3/5$	$1/5$	3

We have completed another full iteration of the simplex algorithm. Now start over.

(1) Find the feasible basis.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
		a_1	a_2	a_3	a_4	a_5	c
0	a_3	0	0	1	$8/5$	$-1/5$	12
-2	a_2	0	1	0	$2/5$	$1/5$	6
1	a_1	1	0	0	$-3/5$	$1/5$	3

(2) Compute the gutter, using $\varphi \cdot a_i - b_i$.

		b_1	b_2	b_3	b_4	b_5	
		-1	-2	0	0	0	
φ		a_1	a_2	a_3	a_4	a_5	c
0	a_3	0	0	1	$8/5$	$-1/5$	12
-2	a_2	0	1	0	$2/5$	$1/5$	6
1	a_1	1	0	0	$-3/5$	$1/5$	3
		0	0	0	$-1/5$	$-3/5$	-15

Now a_2, a_2 appears in the basis. This means $x = [3, 6, 12, 0, 0]$ is a feasible solution (with value -15). Projecting back to $\mathbb{R}^2$, we are at the extreme point $x = (3, 6)$ of the feasible set; see Figure 1.

(3) Find the largest positive entry in the gutter. There is none! Algorithm terminates with unique solution $x = (3, 6)$ and value 15. NOTE: the solution to the original problem is $+15$, not -15. This is because it was originally a maximizing problem.

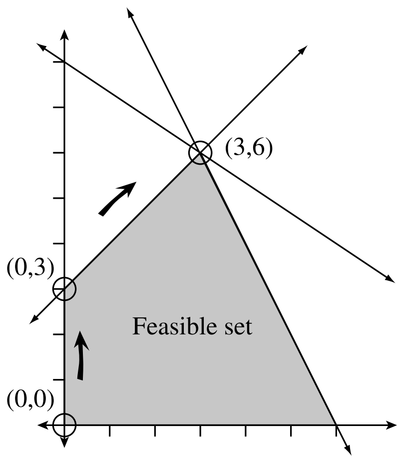

FIGURE 1. Moving through the extreme points via simplex algorithm.

2. 2-Phase Simplex Algorithm (Artificial Basis)

Solve via the simplex algorithm:

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & & 10 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

?????

New sample:

$$\begin{array}{rrcl} \text{minimize} & -x_1 - 2x_2 - 3x_3 + x_4 & & \\ \text{subject to} & x_1 + 2x_2 + 3x_3 & = & 15 \\ & 2x_1 + x_2 + 5x_3 & = & 20 \\ & x_1 + 2x_2 + x_3 + x_4 & = & 10 \\ & x_i & \geq & 0. \end{array}$$

The simplex tableau (with no initial basis) is:

b_1	b_2	b_3	b_4	
-1	-2	-3	1	
a_1	a_2	a_3	a_4	c
1	2	3	0	15
2	1	5	0	20
1	2	1	1	10

There is no obvious choice for a basis, but we already have equalities, so we can't just add slack variables like earlier. a_4 is a good choice for a basis vector. We'll add two *artificial* basis vectors and augment the objective function. We put the new stuff on the right so it can easily be discarded once we get a feasible basis for the original problem.

b_1	b_2	b_3	b_4		b_5	b_6
-1	-2	-3	1		w	w
a_1	a_2	a_3	a_4	c	a_5	a_6
1	2	3	0	15	1	0
2	1	5	0	20	0	1
1	2	1	1	10	0	0

The coefficients of the artificial basis vectors a_5, a_6 is some super-big number $w \gg 0$. Hopefully, this will allow all artificial basis vectors to be replaced by vectors with smaller coefficients. If all artificial vectors can be replaced, the resulting basis is feasible for the original problem. If not, then the original problem is not feasible.

(1) Find the feasible basis, compute the gutter, locate pivot.

		b_1	b_2	b_3	b_4		b_5	b_6		
		-1	-2	-3	1		w	w		
		a_1	a_2	a_3	a_4	c	a_5	a_6		
w	a_5	1	2	3	0	15	1	0	$15/3 = 5$	
w	a_6	2	1	(5)	0	20	0	1	$20/5 = 4$	$\Leftarrow$
1	a_4	1	2	1	1	10	0	0	$10/1 = 10$	
		$3w$	$3w$	$8w$	0	$35w$	0	0		
				$\Uparrow$						

(2) Use row-reduction to make a_3 into $\begin{bmatrix}0\\1\\0\end{bmatrix}$. Then go through the steps again.

		b_1	b_2	b_3	b_4		b_5	b_6		
		-1	-2	-3	1		w	w		
		a_1	a_2	a_3	a_4	c	a_5	a_6		
w	a_5	$-\frac{1}{5}$	($\frac{7}{5}$)	0	0	3	1	-3	$3/(7/5) = 15/7 = 2\frac{1}{7}$	$\Leftarrow$
-3	a_3	$\frac{2}{5}$	$\frac{1}{5}$	1	0	4	0	$\frac{1}{5}$	$4/(1/5) = 20$	
1	a_4	$\frac{3}{5}$	$\frac{9}{5}$	0	1	6	0	$-\frac{1}{5}$	$6/(9/5) = 10/3 = 3\frac{1}{3}$	
		$-w$	$\frac{7}{5}w$	0	0	$3w$	0	$-w$		
			$\Uparrow$							

(3) Use row-reduction to make a_2 into $\begin{bmatrix}1\\0\\0\end{bmatrix}$. Then go through the steps again.

		b_1	b_2	b_3	b_4		b_5			
		-1	-2	-3	1		w			
		a_1	a_2	a_3	a_4	c	a_5	a_6		
-2	a_2	$-\frac{1}{7}$	1	0	0	3	$\frac{5}{7}$		•	
-3	a_3	$\frac{3}{7}$	0	1	0	4	$-\frac{1}{5}$		$\frac{25}{7}/\frac{3}{7} = 8\frac{1}{3}$	
1	a_4	($\frac{6}{7}$)	0	0	1	6	$-\frac{9}{5}$		$\frac{15}{7}/\frac{6}{7} = 2\frac{1}{2}$	$\Leftarrow$
		$\frac{6}{7}$	0	0	0	$-\frac{90}{7}$	$-\frac{92}{35} - w$			
		$\Uparrow$								

Now the artificial basis vectors have all been replaced, so we can discard the junk at the right. This means we have obtained an initial feasible basis for the original problem, i.e., $y = \left[0, \frac{15}{7}, \frac{25}{7}, \frac{15}{7}\right]^\top$ is a feasible initial solution to the original.

But that doesn't mean we're finished. We still have a pivot.

(4) Use row-reduction to make a_1 into $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

		b_1	b_2	b_3	b_4	
		-1	-2	-3	1	
		a_1	a_2	a_3	a_4	c
-2	a_2	0	1	0	$\frac{1}{6}$	$\frac{5}{2}$
-3	a_3	0	0	1	$-\frac{3}{7}$	$\frac{5}{2}$
1	a_1	1	0	0	$\frac{7}{6}$	$\frac{5}{2}$
		0	0	0	$\bullet$	-15

No gutter entries are positive! Algorithm terminates with optimal solution

$$y = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{bmatrix} \qquad \text{with minimal value } \; -15.$$

3. Issues with the Simplex Algorithm

Sometimes things don't go as expected.

3.1. **Unbounded values.**

Consider

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & \tfrac{1}{2}x_1 - x_2 \leq 1 \\ & x_i \geq 0. \end{aligned}$$

Convert to standard form:

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 = 1 \\ & \tfrac{1}{2}x_1 - x_2 + x_4 = 1 \\ & x_i \geq 0. \end{aligned}$$

Now apply the Simplex Algorithm.

(1) Find the feasible basis, compute the gutter, locate pivot.

		b_1	b_2	b_3	b_4		
		-2	-1	0	0		
		a_1	a_2	a_3	a_4	c	
0	a_3	-1	1	1	0	1	$\bullet$
0	a_4	$\textcircled{\frac{1}{2}}$	-1	0	1	1	$\Leftarrow$
		2	1	0	0	0	
		$\Uparrow$					

(2) Row-reduce. Then find the feasible basis, compute the gutter, locate pivot.

		b_1	b_2	b_3	b_4			
		-2	-1	0	0			
		a_1	a_2	a_3	a_4	c		
0	a_3	0	-1	1	2	3	$\bullet$	?
-2	a_1	1	-2	0	2	2	$\bullet$	?
		0	5	0	-4	-4		
			$\Uparrow$					

There is a vector for which improvement of the value is available, but there is no adjacent basic feasible solution which will achieve this improvement. Consequently, the value of the problem is unbounded. Note that the feasible set being unbounded does not force the value to be unbounded. E.g., consider

$$\text{maximize} \quad -x_1$$

on the same feasible set as the above example. This has an optimal solution.

3.2. **Nonunique maximizer.**

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & x_i \geq 0. \end{aligned}$$

Convert to standard form:

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0. \end{aligned}$$

Now apply the Simplex Algorithm.

(1) Find the feasible basis, compute the gutter, locate pivot.

		b_1	b_2	b_3		
		-1	-1	0		
		a_1	a_2	a_3	c	
0	a_3	(1)	(1)	1	1	$\Leftarrow$
		1	1	0	0	
		$\Uparrow$	$\Uparrow$			

Which pivot to pick? There are two equal and optimal choices in the gutter.

(2) Use row-reduction to make a_1 into $[\,1\,]$. Okay, it already is. Then go through the steps again.

		b_1	b_2	b_3	
		-1	-1	0	
		a_1	a_2	a_3	c
-1	a_1	1	1	1	1
		0	0	-1	-1

Terminates with optimal solution

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{with value } \; -1.$$

Now consider the other choice:

		b_1	b_2	b_3	
		-1	-1	0	
		a_1	a_2	a_3	c
-1	a_2	1	1	1	1
		0	0	-1	-1

Terminates with optimal solution

$$y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{with value } \; -1.$$

General solution is (all with value 1 for the original problem):

$$y = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad 0 \le \alpha \le 1.$$

Whenever the algorithm terminates with more 0's than basis vectors (like in this example), look for more solutions!