Classic Optimization Theory and Application

REVIEW QUESTIONS FOR FINAL

1. Polytopes & polyhedra
   (a) Not convex, so neither.

   (b) This is the intersection of two half-spaces, so polytope. But not bounded, so not polyhedron.

   (c) Cannot be realized as a *finite* intersection of half-spaces, so neither.

   (d) Both; it's the bounded intersection of 5 half-spaces.

   (e) Neither; polytopes and polyhedra are closed sets by definition.

   (f) Ill-defined question. An equilateral triangle is a star-shaped region for which the answer is yes to both. However, if "star-shaped" is meant to imply not convex, then no to both.

2. Convexity
   (a) Convex.

   (b) Not convex — the line from -1 to 1 has endpoints in the set but is not contained in the set, since $0 = (0.5)1 + (0.5)(-1)$ is not contained.

   (c) Convex.

3. More convexity
   (a) This is given as the intersection of three half-spaces, so it better be convex!

   (b) This is the axes in $\mathbb{R}^2$ which is not convex — try the line segment from $(0, 2)$ to $(2, 0)$.

(c) This is not convex; it's an annulus. Try the line segment from $(-1, 0)$ to $(1, 0)$.

(d) This is the intersection of two half-spaces, so convex.

(e) This is the intersection of the previous problem with a non-convex set (similar to 1(a)). In general, this need not imply anything about the intersection (eg, intersect 2(c) with quadrant 1). However, in this case the intersection is not convex; try the segment from $(0, 1)$ to $(1, 0)$.

4. Convex hulls

(a) Square centered at origin, sides parallel to axes.

(b) Isosceles triangle "pointing away from origin".

(c) Same as prev.

(d) Unit disk.

(e) $\mathbb{R}^2$.

(f) "Unit diamond" — square rotated 45°.

5. Faces and extreme points.

(a) Δ_1 is the segment from $(0, 1)$ to $(1, 0)$, which are its extreme points. It has one 1-dimensional face; itself.
Δ_2 is the triangle with vertices (extreme points) $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. It has one 2-dimensional face (the triangle itself) and three 1-dimensional faces (its edges).

(b) etc.

6. Sketch the sets.

(a)

7. Convex functions.

(a) Yes.

(b) No.

(c) Of course! It's affine!

(d) *ibid.*

(e) No.

8. Maxima on convex sets.

   (a)

9. Direction of max ascent/descent.
   This will be ∇f and $-\nabla f$, so

$$\begin{aligned}\nabla f &= \left[\tfrac{\partial}{\partial x}\arctan(x^2+y^2), \tfrac{\partial}{\partial y}\arctan(x^2+y^2)\right] \\ &= \left[\frac{2x}{1+(x^2+y^2)^2}, \frac{2y}{1+(x^2+y^2)^2}\right]\end{aligned}$$

   Then the steepest ascent will be in the direction of

$$\begin{aligned}\nabla f(3,4) &= \left[\frac{2\cdot 3}{1+(3^2+4^2)^2}, \frac{2\cdot 4}{1+(3^2+4^2)^2}\right] \\ &= \left[\frac{6}{626}, \frac{8}{626}\right] \\ &= \left[\frac{3}{313}, \frac{4}{313}\right],\end{aligned}$$

   which is also in the direction of $(3,4)$, the given data. This makes sense, as f is a radial function, and increasing on $\mathbb{R}^+$.

10. Find all critical points.

   (a) $f(x,y) = \exp(x^2+y^2)$.

$$\nabla f = [2x, 2y]\exp(x^2+y^2) = 0 \qquad \Longrightarrow \qquad (0,0).$$

   (b) $f(x,y,z) = \arctan(1+x^2+y^2+z^2)$.

$$\nabla f = [2x, 2y, 2z]\frac{1}{1+(1+x^2+y^2+z^2)^2} = 0 \qquad \Longrightarrow \qquad (0,0,0).$$

(c) $f(x, y, z) = x^2 - y^2 + z^2 + 2xy - xz - yz + x - y + z$.

$$\nabla f = [2x + 2y - z + 1, 2x - 2y - z - 1, 2z - x - y + 1].$$

(d) $f(\boldsymbol{x}) = \frac{1}{1+\|\boldsymbol{x}\|^2}$. $\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$, so compute

$$\frac{\partial f}{\partial x_i} = \frac{-\frac{\partial}{\partial x_i}\left(\|\boldsymbol{x}\|^2\right)}{\left(1 + \|\boldsymbol{x}\|^2\right)^2} = \frac{-\frac{\partial}{\partial x_i}\left(x_1^2 + \cdots + x_n^2\right)}{\left(1 + \|\boldsymbol{x}\|^2\right)^2} = -\frac{2x_i}{\left(1 + \|\boldsymbol{x}\|^2\right)^2}$$

Thus,

$$\nabla f = [-2x_1, \ldots, -2x_n]\frac{1}{\left(1+\|\boldsymbol{x}\|^2\right)^2} = -\frac{2\boldsymbol{x}}{\left(1+\|\boldsymbol{x}\|^2\right)^2},$$

which is 0 iff $\boldsymbol{x} = 0$.

(e) $f(\boldsymbol{x}) = \sin(\boldsymbol{x} \cdot \boldsymbol{x})$. $\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$, so compute

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \frac{\partial}{\partial x_i}\sin(x_1^2 + \cdots + x_n^2)\\ &= \cos(x_1^2 + \cdots + x_n^2)\cdot(2x_i)\\ &= 2x_i\cos(\boldsymbol{x}\cdot\boldsymbol{x})\end{aligned}$$

Thus,

$$\nabla f = [2x_1, \ldots, 2x_n]\cos(\boldsymbol{x}\cdot\boldsymbol{x}) = 2\boldsymbol{x}\cos(\boldsymbol{x}\cdot\boldsymbol{x}),$$

which is 0 iff $\boldsymbol{x} = 0$ or $\boldsymbol{x}\cdot\boldsymbol{x} = \frac{2k+1}{2}\pi$. We have $\boldsymbol{x}\cdot\boldsymbol{x} = \frac{2k+1}{2}\pi$ exactly when $\boldsymbol{x}$ lies on the n-sphere of radius $\sqrt{\frac{2k+1}{2}\pi}$

11. Find all critical points.

$$\nabla f = [2x + y - 1, -2y + x - 1]$$

Setting this to 0 gives two linear equations; solving them gives $(x, y) = (-\frac{3}{5}, \frac{1}{5})$.

$$H_f(x, y) = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \quad \Longrightarrow \quad H_f\left(-\tfrac{3}{5}, \tfrac{1}{5}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

This has eigenvalues $\pm\sqrt{5}$, so it is a saddle (not local or global extreme point).

12. Find all critical points.

$$\nabla f = [yz + 2x, xz + 2y, xy + 2z]$$

Setting this to 0 gives three (nonlinear) equations. First notice that if one variable is 0, all must be 0 to solve the system. This means $(0, 0, 0)$ is a critical point, and for the rest of the time, we can assume $x, y, z \neq 0$. Solving them gives four more critical points $(x, y, z) = (\pm 2, \mp 2, 2), (\pm 2, \pm 2, -2)$. However, none of these are within the constraint, except $(0, 0, 0)$.

$$H_f(x, y, z) = \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix} \qquad \Longrightarrow \qquad H_f(0, 0, 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This is positive definite, so it is a local min. Since f is convex within the constraint set, this is also a global min.

13. Use Lagrange multipliers to find the maximum value of $f(x, y) = 3x + 4y$ s.t. the constraint $x^2 + 4y^2 = 1$.
The constraint function is $g(x, y) = x^2 + 4y^2 - 1$ and the Lagrangian gives

$$\nabla f = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \lambda 2x \\ \lambda 8y \end{bmatrix} = \lambda \nabla g.$$

This gives

$$x = \tfrac{3}{2\lambda}, \quad y = \tfrac{4}{8\lambda} = \tfrac{1}{2\lambda},$$

which we put into the constraint to get

$$\begin{aligned} \left(\tfrac{3}{2\lambda}\right)^2 + 4\left(\tfrac{1}{2\lambda}\right)^2 &= 1 \\ \tfrac{9}{4\lambda^2} + \tfrac{4}{4\lambda^2} &= 1 \\ 13 &= 4\lambda^2 \\ \lambda &= \pm\tfrac{\sqrt{13}}{2} \end{aligned}$$

Thus,

$$x = \pm\tfrac{3}{\sqrt{13}}, \quad y = \pm\tfrac{1}{\sqrt{13}},$$

and the max and min are

$$f\left(\tfrac{3}{\sqrt{13}}, \tfrac{1}{\sqrt{13}}\right) = \tfrac{13}{\sqrt{13}} = \sqrt{13}, \quad \text{and} \quad f\left(-\tfrac{3}{\sqrt{13}}, -\tfrac{1}{\sqrt{13}}\right) = -\sqrt{13}.$$

14. Find the max and min of $f(x, y, z) = x - y - z$ s.t.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x + y + z &= 0. \end{aligned}$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$\begin{aligned} g_1(x, y, z) &= x^2 + y^2 + z^2 - 6 \\ g_2(x, y, z) &= x + y + z. \end{aligned}$$

$$\nabla f = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 2x + \lambda_2 \\ \lambda_1 2y + \lambda_2 \\ \lambda_1 2z + \lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

Working each eqn individually gives

$$x = \tfrac{1-\lambda_2}{2\lambda_1}, \quad y = \tfrac{-1-\lambda_2}{2\lambda_1}, \quad z = \tfrac{-1-\lambda_2}{2\lambda_1} = y,$$

so the second constraint gives

$$x + y + z = x + 2y = \tfrac{-1-3\lambda_2}{2\lambda_1} = 0 \implies \lambda_2 = -\tfrac{1}{3},$$

so

$$x = \tfrac{4/3}{2\lambda_1} = \tfrac{2}{3\lambda_1}, \quad y = z = \tfrac{-2/3}{2\lambda_1} = -\tfrac{1}{3\lambda_1}.$$

Putting into the first constraint,

$$\begin{aligned} \left(\tfrac{2}{3\lambda_1}\right)^2 + 2\left(\tfrac{1}{3\lambda_1}\right)^2 &= 6 \\ \tfrac{4}{9\lambda_1^2} + \tfrac{2}{9\lambda_1^2} &= 6 \\ \lambda_1^2 &= \tfrac{1}{9} \\ \lambda &= \pm\tfrac{1}{3}. \end{aligned}$$

Substituting into the eqs for x, y, z:

$$x = \tfrac{2}{3\lambda_1} = \pm 2, \quad y = z = -\tfrac{1}{3\lambda_1} = \mp 1.$$

Thus the max and min are

$$f(2,-1,-1) = 4 \quad \text{and} \quad f(-2,1,1) = -4.$$

15. The planes $x + y - z - 2w = 1$ and $x - y + z + 2w = 2$ intersect in a set $F \subseteq \mathbb{R}^4$. Find the point in F that is nearest to the origin.
Minimize (distance)^2 to the origin to minimize distance to the origin; i.e., minimize $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2$ s.t.

$$\begin{aligned} x + y - z - 2w &= 1 \\ x - y + z + 2w &= 2 \end{aligned}$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$\begin{aligned} g_1(x,y,z,w) &= x + y - z - 2w - 1 \\ g_2(x,y,z,w) &= x - y + z + 2w - 2. \end{aligned}$$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \\ 2w \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \\ -\lambda_1 + \lambda_2 \\ -2\lambda_1 + 2\lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

The first two together give

$$x + y = \lambda_1, \quad x - y = \lambda_2,$$

and the second two give

$$z + 2w = -\tfrac{5}{2}(\lambda_1 - \lambda_2).$$

Putting these into the constraints (with $g_1 = 0, g_2 = 0$),

$$\lambda_1 + \tfrac{5}{2}(\lambda_1 - \lambda_2) = 1$$
$$\lambda_2 - \tfrac{5}{2}(\lambda_1 - \lambda_2) = 2.$$

Solving gives $\lambda_1 = \frac{17}{12}, \lambda_2 = \frac{19}{12}$. Then $x = \frac{3}{2}, y = -\frac{1}{12}$, so we obtain the new constraint $z + 2w = \frac{5}{12}$. This line is the set F, and it allows us to reduce f to a function of one variable:

$$\begin{aligned} f(x, y, z, w) &= f\left(\tfrac{3}{2}, -\tfrac{1}{12}, \tfrac{5}{12} - 2w, w\right) \\ &= f(w) \\ &= \left(\tfrac{3}{2}\right)^2 + \left(\tfrac{1}{12}\right)^2 + \left(\tfrac{5}{12} - 2w\right)^2 + w^2 \\ &= 5w^2 + -\tfrac{5}{3}w + \tfrac{175}{72}, \end{aligned}$$

with crit pt:

$$\begin{aligned} f'(w) &= 10w - \tfrac{5}{3} = 0 \\ w &= \tfrac{1}{6}. \end{aligned}$$

Thus, the closest point of F to the origin is $\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right)$. Since $f\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right) = \frac{55}{24}$, it is a distance of $\sqrt{\frac{55}{24}}$ away.

16. Use an appropriate minimum problem to show that

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}, \qquad a, b, c > 0.$$

Define $f(x, y, z) = \sqrt[3]{xyz} - \frac{x+y+z}{3}$. We must show $\max_{x,y,z>0} f(x, y, z) \leq 0$. So we are in the first quadrant. Crit pts:

$$\nabla f = \begin{bmatrix} \frac{(yz)^{1/3}}{3x^{1/3}} \\ \frac{(xz)^{1/3}}{3y^{1/3}} \\ \frac{(xy)^{1/3}}{3z^{1/3}} \end{bmatrix} \qquad \text{cannot be 0 for } x, y, z > 0 \text{ (or otherwise).}$$

So look to boundary (for the moment, allow $x, y, z \geq 0$). For $z = 0$, we are in the xy-plane (the "floor") and

$$f(x, y, z) = -\tfrac{(x+y)}{3} \leq 0 \quad \text{since } x, y \geq 0.$$

This will only become larger when x, y get closer to 0, so

$$\max_{z=0} f(x,y,z) = f(0,0,0) = 0.$$

The situation is similar/symmetric for the other two planes. Therefore,

$$\max_{x,y,z>0} f(x,y,z) < 0.$$

17. Use an appropriate minimum problem to show that

$$(a_1 a_2 \dots a_n)^{1/n} \leq \tfrac{1}{n}(a_1 a_2 \dots a_n), \qquad a_i > 0.$$

Same soln as previous problem.

18. Find the minimum and maximum value of $f(x,y,z) = x + 2y + 3z$. in the domain $D = \{3x^2 + 2y^2 + z^2 \leq 1\}$.
Constraint function is $g(x,y,z) = 3x^2 + 2y^2 + z^2 - 1$.

case (i) $g < 0$. Then $\lambda = 0$ and $\nabla f = 0$, so check crit pts.

$$\begin{aligned} 1 &= 0 \\ 2 &= 0 \quad \Longrightarrow ?? \\ 3 &= 0 \end{aligned}$$

The partials of a linear function cannot simultaneously vanish unless it is a constant function! So we have no critical points; the extrema of a linear function always occur on the boundary.

case (ii) $g = 0$. Then $\lambda \neq 0$ and we check the Lagrangian.

$$\begin{aligned} 1 &= -\lambda 6x \\ 2 &= -\lambda 4y \\ 3 &= -\lambda 2z \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= -\tfrac{1}{6\lambda} \\ y &= -\tfrac{1}{4\lambda} \\ z &= -\tfrac{1}{2\lambda} \end{aligned}$$

Then use the constraint

$$\begin{aligned} \tfrac{3}{36\lambda^2} + \tfrac{2}{16\lambda^2} + \tfrac{1}{4\lambda^2} &= 1 \\ 11 &= 24\lambda^2 \\ \lambda &= \pm\sqrt{\tfrac{11}{24}} \\ \tfrac{1}{\lambda} &= \pm 2\sqrt{\tfrac{6}{11}} \end{aligned}$$

This gives $x = \mp\frac{1}{3}\sqrt{\frac{6}{11}}, y = \mp\frac{1}{2}\sqrt{\frac{6}{11}}, z = \mp\sqrt{\frac{6}{11}}$, so

$$f(x_+, y_+, z_+) = 13\sqrt{\tfrac{2}{33}} \quad \text{and}$$

$$f(x_-, y_-, z_-) = -13\sqrt{\tfrac{2}{33}}$$

are the max and min.

19. This is problem 15 again.

20. Find the max and min of $f(x, y) = 7x + 8y$ under the constraints

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 12 \\ 3x + y &\leq 24. \end{aligned}$$

The feasible region has vertices $(0, 0), (0, 6), (\frac{36}{5}, \frac{12}{5}), (8, 0)$.

$$\begin{aligned} f(0, 0) &= 0 \\ f(0, 6) &= 48 \\ f\left(\tfrac{36}{5}, \tfrac{12}{5}\right) &= \tfrac{348}{5} = 69\tfrac{3}{5} \\ f(8, 0) &= 56 \end{aligned}$$

So the max is at $(\frac{36}{5}, \frac{12}{5})$ and the min is at $(0, 0)$.
Standard form: maximize $f(x, y) = 7x + 8y$ s.t.

$$\begin{aligned} x + 2y + u \qquad &= 12 \\ 3x + y \qquad + v &= 24 \\ x &\geq 0 \\ y &\geq 0 \\ u &\geq 0 \\ v &\geq 0, \end{aligned}$$

so maximize $f(\boldsymbol{x}) = (7, 8, 0, 0) \cdot \boldsymbol{x}$ s.t.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 12 \\ 24 \end{bmatrix}$$

$$\boldsymbol{x} \geq 0$$

$$u, v \geq 0.$$

21. (a) Max $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y &\leq 1 \\ x + y &\geq 1 \end{aligned}$$

$$\begin{aligned} -x+y &\geq 0 \\ 2x+y &\geq 1 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x,y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad\qquad &= 1 \\ -x - y \quad + v \qquad\qquad &= -1 \\ x - y \qquad\quad + w \quad &= 0 \\ -2x - y \qquad\qquad\quad + z &= -1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(x,y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad\qquad &= 1 \\ x + y \quad - v \qquad\qquad &= 1 \\ x - y \qquad\quad + w \quad &= 0 \\ 2x + y \qquad\qquad\quad - z &= 1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(\boldsymbol{x}) = (3, -4, 0, 0, 0, 0) \cdot \boldsymbol{x}$ s.t.

$$\begin{bmatrix} 4 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} \geq 0$$

$$u, v, w, z \geq 0.$$

(b) Minimize $f(x,y,z) = x - y + z$ s.t.

$$\begin{aligned} 4x + y - z &\geq 1 \\ 4x + y - z &\leq 1 \\ x + y + z &= 1 \\ x, y, z &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x,y) = -x + y - z$ s.t.

$$\begin{aligned} 4x + y - z &= 1 \\ x + y + z &= 1 \\ x, y, z &\geq 0, \end{aligned}$$

i.e.,

$$\begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\boldsymbol{x} \geq 0.$$

22. (a)

$$\begin{aligned} x + 2y - z &= 4 \\ -x + 2y - z &= 2 \\ 2y - z &= 3 \\ x, y, z &\geq 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ -1 & 2 & -1 & 2 \\ 0 & 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we have $\left(1, \frac{3}{2} - \frac{z}{2}, z\right)$, or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Then putting $x_3 = 0$ (i.e., using a basis of a_1, a_2) gives

$$x = \begin{bmatrix} 1 \\ 3/2 \\ 0 \end{bmatrix}.$$

Putting $x_2 = 0$ (i.e., using a basis of a_1, a_3) gives

$$x = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

Putting $x_1 = 0$ leaves something which is not solvable, so we only have 2 basic solutions, and the second one isn't feasible.

(b)

$$\begin{aligned} w - x + y - z &= 2 \\ w + x + y + z &= 3 \\ w, x, y, z &\geq 0 \end{aligned}$$

Reducing the system gives

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 5/2 \\ 0 & 1 & 0 & 1 & 1/2 \end{bmatrix}.$$

Denoting the solution obtained from basis a_i, a_j by x_{ij}, we get

$$x_{12} = \begin{bmatrix} 5/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \quad x_{14} = \begin{bmatrix} 5/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \quad x_{23} = \begin{bmatrix} 0 \\ 1/2 \\ 5/2 \\ 0 \end{bmatrix}, \quad x_{34} = \begin{bmatrix} 0 \\ 0 \\ 5/2 \\ 1/2 \end{bmatrix}.$$

(c)

$$\begin{aligned} w - x + y - z &= 2 \\ w + x + y + z &= 3 \\ w - x - y + z &= 4 \\ w, x, y, z &\geq 0 \end{aligned}$$

Reducing the system gives

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & 1 & 4 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & -2 & 2 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 7/2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \end{aligned}$$

Denoting the solution obtained from basis a_i, a_j, a_k by x_{ijk}, we get

$$x_{123} = \begin{bmatrix} 7/2 \\ 1/2 \\ -1 \end{bmatrix}, \quad x_{124} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix}, \quad x_{134} = \begin{bmatrix} 3 \\ -1/2 \\ -1 \end{bmatrix}, \quad x_{234} = \begin{bmatrix} -3 \\ 5/2 \\ 7/2 \end{bmatrix}.$$

23. ([Deleted]) Using the method of basic solutions, find the extreme points of the following convex sets.

(a)

$$\begin{aligned} 4x - y &\leq 1 \\ x + y &\geq 1 \\ -x + y &\geq 0 \\ 2x + y &\geq 1 \end{aligned}$$

First, put the system into standard form as

$$\begin{aligned} 4x - y + u \quad\quad\quad\quad &= 1 \\ x + y \quad - v \quad\quad\quad &= 1 \\ -x + y \quad\quad - w \quad &= 0 \\ 2x + y \quad\quad\quad - z &= 1 \end{aligned}$$

with $u, v, w, z \geq 0$. Then use Gaussian elimination:

$$\begin{bmatrix} 4 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 1 & 0 & 0 & -2/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/3 & -2/3 & -1/3 \end{bmatrix}$$

Denoting the solution obtained from basis a_i, a_j, a_k, a_ℓ by $x_{ijk\ell}$, we get

$$x_{1234} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix}, etc.$$

(b) Combining the first two, we get

$$\begin{aligned} 4x + y - z &= 1 \\ x + y + z &= 1 \end{aligned}$$

which, in matrices, reduces to

$$\begin{bmatrix} 4 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 5/3 & 1 \end{bmatrix}$$

This gives

$$x_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; x_{13} = \begin{bmatrix} 2/5 \\ 0 \\ 3/5 \end{bmatrix}, \; x_{23} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

24. Consider the system

$$\begin{aligned} 5x + 3y - z - w &= 1 \\ 5x + 4y + z - 2w &= 9 \end{aligned}$$

(a) Find the general solution.

$$\begin{bmatrix} 5 & 3 & -1 & -1 & 1 \\ 5 & 4 & 1 & -2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{7}{8} & \frac{1}{4} & -\frac{23}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{3}{4} & \frac{41}{8} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 - \frac{7}{8}s + \frac{1}{4}t \\ 2 - \frac{9}{8}s - \frac{3}{4}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{7}{8} \\ -\frac{9}{8} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix}$$

(b) Find all solutions satisfying $x, y, z, w \geq 0$.
From the previous part, if $s = 0$, then must have $t \leq \frac{8}{3}$ or else y becomes negative. If $t = 0$, then must have $s \leq \frac{16}{9}$ or else y becomes negative.

(c) Find all basic solutions satisfying $x, y, z, w \geq 0$.
Denoting the solution obtained from basis a_i, a_j, a_k by x_{ijk}, we get

$$x_{12} = \begin{bmatrix} -23/8 \\ -41/8 \end{bmatrix}, x_{13} = \begin{bmatrix} -247/36 \\ -41/9 \end{bmatrix}, x_{14} = \begin{bmatrix} -7/6 \\ -41/6 \end{bmatrix},$$

$$x_{23} = \begin{bmatrix} 247/28 \\ 23/7 \end{bmatrix}, x_{24} = \begin{bmatrix} -7/2 \\ -23/2 \end{bmatrix}, x_{34} = \begin{bmatrix} 14/15 \\ -247/30 \end{bmatrix}.$$

(d) Describe the set of all solutions satisfying $x, y, z, w \geq 0$ geometrically and find all extreme points of the set.
It's a hexagon with extreme points as seen in the previous part.

25. Show that this problem has no solution:

$$\text{Maximize } f(x, y, z, w) = x + y + z + w$$

with respect to

$$\begin{aligned} 5x + 3y - z - w &= 1 \\ 4x + 4y + z - 2w &= 9 \end{aligned}$$

Here the feasible set is $\mathbb{R}^4$, and the solution is a 2-plane. Thus we can make f as big as we want. For example, fix $s = 0$ so we are on the line

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 + \frac{1}{4}t \\ 2 - \frac{3}{4}t \\ 0 \\ t \end{bmatrix}$$

On this line,

$$f(t) = \left(3 + \tfrac{1}{4}t\right) + \left(2 - \tfrac{3}{4}t\right) + t = 5 + \tfrac{1}{2}t.$$

Head off to infinity on this line and $\lim_{t\to\infty} f(t) = \infty$.

26. One solution to

$$\text{Maximize } f(x, y) = x + 2y$$

subject to

$$\begin{aligned} y + z + w &= 3 \\ x + y \qquad - w &= 1 \end{aligned}$$

is given by $(x, y, z, w) = (2, 1, 0, 2)$. Find a basic feasible solution that also solves the problem.

27. Maximize $f(x_1, x_2, x_3) = 2x_1 + 3x_2 + 3x_3$ subject to

$$\begin{aligned} 3x_1 + 2x_2 \qquad\quad &\le 60 \\ -x_1 + x_2 + 4x_3 &\le 10 \\ 2x_1 - 2x_2 + 5x_3 &\le 50 \\ x_i &\ge 0. \end{aligned}$$

Simplex algorithm yields maximum $f(8, 18, 0) = 70$. First, put it into computational/standard form:

$$\begin{array}{ll} \min & -2x_1 - 3x_2 - 3x_3 \\ \text{s/t} & 3x_1 + 2x_2 \qquad + x_4 \qquad\qquad = 60 \\ & -x_1 + x_2 + 4x_3 \qquad + x_5 \qquad = 10 \\ & 2x_1 - 2x_2 + 5x_3 \qquad\qquad + x_6 = 50 \\ & \qquad\qquad\qquad\qquad\qquad x_i \ge 0. \end{array}$$

Now crunch away:

		b_1	b_2	b_3	b_4	b_5	b_6			
		-2	-3	-3	0	0	0			
		a_1	a_2	a_3	a_4	a_5	a_6	c		
0	a_4	3	2	0	1	0	0	60	30	
0	a_5	-1	(1)	4	0	1	0	10	10	$\Leftarrow$
0	a_6	2	-2	5	0	0	1	50	$\bullet$	
		2	3	3	0	0	0	0		
			$\Uparrow$							

		b_1	b_2	b_3	b_4	b_5	b_6			
		-2	-3	-3	0	0	0			
		a_1	a_2	a_3	a_4	a_5	a_6	c		
0	a_4	(5)	0	-8	1	-2	0	40	8	$\Leftarrow$
-3	a_2	-1	1	4	0	1	0	10	$\bullet$	
0	a_6	0	0	11	0	2	1	70	x	
		5	0	-4	0	-3	0	-30		
		$\Uparrow$								

		b_1	b_2	b_3	b_4	b_5	b_6	
		-2	-3	-3	0	0	0	
		a_1	a_2	a_3	a_4	a_5	a_6	c
-2	a_1	1	0	$-8/5$	$1/5$	$-2/5$	0	8
-3	a_2	0	1	$12/5$	$1/5$	$3/5$	0	18
0	a_6	0	0	11	0	2	1	70
		0	0	-1	-1	-1	0	-70

All gutter entries are negative, so algorithm terminates with value 70 (for the original problem) at $(8, 18, 0)$. The 0 is because a_3 doesn't appear in the solution (a_6 gets ignored).

However, you could also have picked a_3 as your basis vector back in the first step. This leads to a different solution. I didn't have time to write this part up (the numbers get ugly and I think I made some arithmetical mistake). So the full solution is any linear combination of these two. Perhaps it leads to the same solution again, but it didn't look like it.

28. Show that the function $f(x_1, x_2, x_3, x_4) = x_3 - x_4$ has no min on the set

$$\begin{aligned} x_1 \qquad - x_4 &= 5 \\ x_2 + 2x_3 \qquad &= 10 \\ x_i &\geq 0. \end{aligned}$$

Fix $x_2 = 4, x_3 = 1$ to make life simpler. Then for

$$f(t) = f(5 + t, 4, 1, t) = 1 - t,$$

both constraints are satisfied for all $t \in \mathbb{R}$ and $\lim_{t\to\infty} f(t) = \infty$.

29. ([Deleted!])

30. ([Deleted!])

31. Find a basic feasible solution of

$$\begin{aligned} x_1 - x_2 \quad\quad &= 1 \\ 2x_1 + x_2 - x_3 &= 3 \\ x_i &\geq 0. \end{aligned}$$

First reduce the system:

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & -1/3 & 1/3 \end{bmatrix}$$

Denoting the solution obtained from basis a_i, a_j by x_{ij}, we get

$$x_{12} = \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}, x_{13} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_{23} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

x_{12} is the only feasible one (the others have negative coords).

32. ([Deleted!])

33. ([Deleted!])

34. Primal:

$$\text{Maximize } 2x_1 + 3x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ 2x_1 \ + x_2 &\leq 5 \\ x_i &\geq 0. \end{aligned}$$

To find the dual, consider

$$\begin{aligned} \boldsymbol{a} &= [2, 3] \\ \boldsymbol{b} &= [4, 5] \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Then the dual is found by

$$\begin{array}{rl} \max & f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} \\ \text{s/t} & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{array} \implies \begin{array}{rl} \min & g(\boldsymbol{y}) = \boldsymbol{b} \cdot \boldsymbol{y} \\ \text{s/t} & A^T\boldsymbol{y} \ge \boldsymbol{a} \\ & \boldsymbol{y} \ge 0 \end{array}$$

and is therefore:

$$\text{Minimize } 4y_1 + 5y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\ge 2 \\ 2y_1 + y_2 &\ge 3 \\ y_i &\ge 0. \end{aligned}$$

Note $A^T = A$ in this example.

35. Consider the linear programming problem

$$\begin{array}{rl} \min & \boldsymbol{a} \cdot \boldsymbol{x} \\ \text{s/t} & A\boldsymbol{x} \ge \boldsymbol{b} \\ & \boldsymbol{x} \ge 0. \end{array}$$

(a) Find the dual.

$$\begin{array}{rl} \max & \boldsymbol{b} \cdot \boldsymbol{y} \\ \text{s/t} & A^T\boldsymbol{y} \le \boldsymbol{a} \\ & \boldsymbol{y} \ge 0. \end{array}$$

(b) Find the dual of the dual.

$$\begin{array}{rl} \min & \boldsymbol{a} \cdot \boldsymbol{x} \\ \text{s/t} & (A^T)^T\boldsymbol{x} \ge \boldsymbol{b} \\ & \boldsymbol{x} \ge 0. \end{array}$$

(c) Show that the dual of the dual is the primal.
$(A^T)^T = A$.

36. ([Deleted!])

37. ([Deleted!])