Classic Optimization Theory and Application

REVIEW QUESTIONS

1. Find the minimum distance to the origin for each affine hyperplane.

(a) The line in $\mathbb{R}^2$ given by $3x - 4y = 5$.
Let $f(x,y) = x^2 + y^2$. Since

$$-4y = 5 - 3x \implies y = \tfrac{3}{4}x - \tfrac{5}{4},$$

we can rewrite f as a function of x alone:

$$f(x,y) = f\left(x, \tfrac{3}{4}x - \tfrac{5}{4}\right) = x^2 + \left(\tfrac{3}{4}x - \tfrac{5}{4}\right)^2 = \tfrac{25}{16}x^2 - \tfrac{15}{8}x + \tfrac{25}{16}.$$

Then

$$f'(x) = \tfrac{25}{8}x - \tfrac{15}{8} = 0$$

implies $x = \frac{3}{5}, y = -\frac{4}{5}$, and

$$\sqrt{f\left(\tfrac{3}{5}, -\tfrac{4}{5}\right)} = \sqrt{\tfrac{9}{25} + \tfrac{16}{25}} = 1.$$

(b) The line in $\mathbb{R}^2$ given by $4x + 5y + 5 = 0$.
Let $f(x,y) = x^2 + y^2$. Since

$$5y = -4x - 5 \implies y = -\tfrac{4}{5}x - 1,$$

we can rewrite f as a function of x alone:

$$f(x,y) = f\left(x, -\tfrac{4}{5}x - 1\right) = \tfrac{41}{25}x^2 - \tfrac{8}{5}x + 1.$$

Then

$$f'(x) = \tfrac{82}{25}x - \tfrac{8}{5} = 0$$

implies $x = \frac{20}{41}, y = -\frac{57}{41}$, and

$$\sqrt{f\left(\tfrac{20}{41}, -\tfrac{57}{41}\right)} = \sqrt{\tfrac{400}{1681} + \tfrac{3249}{1681}} = \sqrt{\tfrac{3649}{1681}} = \sqrt{\tfrac{89}{41}}.$$

(c) The plane of $\mathbb{R}^3$ given by $2x + 2y - z = 9$.
Let $f(x,y,z) = x^2 + y^2 + z^2$. Since

$$z = 2x + 2y - 9,$$

we can rewrite f as a function of x and y:

$$f(x, y, 2x + 2y - 9) = 5x^2 + 8xy + 5y^2 - 36x - 36y + 81.$$

Then

$$\nabla f(x, y) = \begin{bmatrix} 10x + 8y - 36 \\ 10y + 8x - 36 \end{bmatrix} = 0$$

gives two simultaneous eqns to solve:

$$\begin{aligned} 5x + 4y &= 18 \\ 4x + 5y &= 18 \\ 20x + 16y &= 72 \\ -20x + -25y &= -90 \\ -9y &= -18 \\ y &= 2, x = 2. \end{aligned}$$

Then the distance is

$$\sqrt{f(2, 2, -5)} = \sqrt{33}.$$

(d) The hyperplane in $\mathbb{R}^4$ given by $4w - 4x + 2y - 8z + 170 = 0$.
Let $f(w, x, y, z) = w^2 + x^2 + y^2 + z^2$. Since

$$z = \tfrac{w}{2} - \tfrac{x}{2} + \tfrac{y}{4} + \tfrac{85}{4},$$

we can rewrite f as a function of w, x, y:

$$f\left(w, x, y, \tfrac{w}{2} - \tfrac{x}{2} + \tfrac{y}{4} + \tfrac{85}{4}\right)$$
$$= \tfrac{5w^2}{4} + \tfrac{x^2}{4} + \tfrac{17y^2}{16} - \tfrac{wx}{2} + \tfrac{wy}{4} - \tfrac{xy}{4} + \tfrac{85w}{4} - \tfrac{85x}{4} + \tfrac{85y}{8} + \tfrac{7225}{16}.$$

Then

$$\nabla f(w, x, y) = \begin{bmatrix} \tfrac{5w}{2} - \tfrac{x}{2} + \tfrac{y}{4} + \tfrac{85}{4} \\ -\tfrac{w}{2} + \tfrac{x}{2} - \tfrac{y}{4} - \tfrac{85}{4} \\ \tfrac{w}{4} - \tfrac{x}{4} + \tfrac{17y}{8} + \tfrac{85}{8} \end{bmatrix} = 0$$

gives $w = y = 0$, $x = -\frac{85}{2}$, so $z = \frac{85}{2}$, and the distance is

$$\sqrt{f\left(0, -\tfrac{85}{2}, 0, \tfrac{85}{2}\right)} = \sqrt{2\left(\tfrac{85}{2}\right)^2} = \tfrac{85}{\sqrt{2}}.$$

2. Which of the following sets is convex? Sketch the sets and prove your answers!

(a) $\{[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in \mathbb{R}^2 : 0 \le x, 0 \le y, x + y \le 1\}$.
This triangle is convex with extreme points $(0, 0), (1, 0), (0, 1)$; it is the intersection of three half-planes (which ones?).

(b) $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$.
This set is the x-axis and y-axis together. It is not convex because it contains the point $(0, 2)$ and $(2, 0)$, but not the (mid)point

$$(1, 1) = \tfrac{1}{2}(0, 2) + \tfrac{1}{2}(2, 0).$$

(c) $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$.
This is the annulus of points less than 2 from the origin but more than 1 from the origin. It is not convex because it contains $(0, 1)$ and $(0, -1)$ but not the (mid)point

$$(0, 0) = \tfrac{1}{2}(0, 1) + \tfrac{1}{2}(0, -1).$$

(d) $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 : -1 \leq x - y \leq 1\}$.
This diagonal strip is the intersection of two half-planes (which ones?), so it is clearly convex.

(e) $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 : xy \leq 0, -1 \leq x - y \leq 1\}$.
This is the intersection of the previous region with the second and fourth quadrant; it looks like two right triangles which touch at one point. This is not convex because it contains $(0, 1)$ and $(1, 0)$ but not the (mid)point

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{2}(0, 1) + \tfrac{1}{2}(1, 0)$$

3. Find the convex hulls of the following subsets of $\mathbb{R}^n$. Sketch!

(a) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$.

(b) $\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}\right\}$.

(c) $\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}\right\}$.

(d) $\{\bar{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$.

(e) $\{\bar{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$.

(f) $\{\bar{x} \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}$.

4. Prove: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, and if $C \subseteq \mathbb{R}^n$ is a convex set, then $T(C)$ is a convex set. Similarly, if $D \subseteq \mathbb{R}^m$ is a convex set, then $T^{-1}(D)$ is a convex set.
($\Rightarrow$) Let $C \subseteq \mathbb{R}^n$ be convex. Then any point $x \in C$ may be written as a convex combination:

$$x = \sum_{i=1}^{n} c_i x_i, \quad \text{with } c_i \geq 0 \text{ and } \sum_{i=1}^{n} c_i = 1,$$

where $\{x_i\}$ are extreme points of C. Thus

$$\begin{aligned} T(x) &= T\left(\sum_{i=1}^{n} c_i x_i\right) \\ &= \sum_{i=1}^{n} c_i T(x_i) \end{aligned}$$

shows that $T(x)$ will be a convex combination of the extreme points $\{T(x_i)\}$ of $T(C)$.
($\Leftarrow$) Let $D \subseteq \mathbb{R}^m$ be convex. Then $D = conv D$, which we can represent as the intersection of all half-spaces containing D:

$$D = \bigcap_{H:D\subseteq H} H.$$

Then

$$T^{-1}(D) = T^{-1}\left(\bigcap_{H:D\subseteq H} H\right) = \bigcap_{H:D\subseteq H} T^{-1}(H)$$

5. Sketch the following sets. Which of the sets is convex? closed? bounded? If possible, find all the faces and the extreme points.
   (a) $\{(x, y) : 0 \leq x, y \text{ and } x + y \leq 1\}$
   This triangle is closed, convex, and bounded. Its faces are its sides, and extreme points are its vertices.

   (b) $\{(x, y) : 0 \leq x, y \text{ and } 1 \leq x + y\}$
   This intersection of three half-planes is closed and convex, but not bounded.

(c) $\{(x, y) : 1 \leq x + y \leq 2\}$
This diagonal strip is closed and convex, but not bounded.

(d) $\{(x, y) : 1 \leq x + y \leq 2 \text{ and } 1 \leq x - y \leq 2\}$
This square is closed, convex, and bounded.

(e) $\{(x, y) : 2 \leq x^2 + y^2 \leq 5\}$
This annulus is closed and bounded, but not convex.

(f) $\{(x, y, z) : |x| + |y| + |z| \leq 1\}$
This octahedron is closed, convex, and bounded.

6. Which of the following functions is convex on $\mathbb{R}^2$?

(a) $f(x, y) = x^2 + y^4$.

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix},$$

so

$$H_f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} > 0 \text{ for all } y \neq 0,$$

which shows H_f is convex on $\mathbb{R}^2$.

(b) $f(x, y) = x^3 + y^2$.

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 2y \end{bmatrix},$$

so

$$H_f(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix} > 0 \text{ for all } x > 0,$$

which shows H_f is only convex on regions in the half-plane $\{x > 0\}$.

(c) $f(x, y) = x - y$.
This function is affine, and hence convex everywhere by Thm. 11.

(d) $f(x, y) = ax + by$, where a and b are constants.
This function is affine, and hence convex everywhere by Thm. 11.

(e) $f(x,y) = -\ln(1 + x^2 + y^2)$.

$$\nabla f(x,y) = \begin{bmatrix} -\frac{2x}{1+x^2+y^2} \\ -\frac{2y}{1+x^2+y^2} \end{bmatrix},$$

so

$$H_f(x,y) = \begin{bmatrix} \frac{x^2-y^2-1}{(1+x^2+y^2)^2} & -\frac{4xy}{(1+x^2+y^2)^2} \\ -\frac{4xy}{(1+x^2+y^2)^2} & \frac{-x^2+y^2-1}{(1+x^2+y^2)^2} \end{bmatrix}$$
$$= \frac{1}{(1+x^2+y^2)^2} \begin{bmatrix} x^2-y^2-1 & -4xy \\ -4xy & -x^2+y^2-1 \end{bmatrix}.$$

This has eigenvals

$$\frac{-1 \pm \sqrt{x^4 + 14x^2y^2 + y^4}}{(1+x^2+y^2)^2},$$

so the convexity of f varies. However, it is definitely not convex on all of $\mathbb{R}^2$, e.g.:

$$H_f(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

7. Find all critical points of

$$f(x,y) = x^2 - y^2 + xy - y - x$$

Which of them are actually local extreme values? Which of them are global extreme values?
Check the gradient for crit pts:

$$\nabla f = \begin{bmatrix} 2x + y - 1 \\ -2y + x - 1 \end{bmatrix} = 0.$$

So solve

$$\begin{aligned} 2x + y &= 1 \\ x - 2y &= 1. \end{aligned}$$

$$\begin{aligned} 2x + y &= 1 \\ -2x + 4y &= -2 \\ 5y &= -1 \\ y &= -\tfrac{1}{5}, x = \tfrac{3}{5} \end{aligned}$$

The Hessian is

$$H_f(x, y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

which is constant and has eigenvals $1, 3$ for any value of x, y. Thus, the function is convex with unique (local and global) minimum

$$f\left(\tfrac{3}{5}, -\tfrac{1}{5}\right) = -\tfrac{1}{5}$$

and no maximum.

8. Find all critical points of $f(x, y, z) = xyz + x^2 + y^2 + z^2$ s.t.

$$x + y + z < 2.$$

Which of them are actually local extreme values? Which of them are global extreme values?
Check the gradient for crit pts:

$$\nabla f = \begin{bmatrix} yz + 2x \\ xz + 2y \\ xy + 2z \end{bmatrix} = 0.$$

Clearly, $(0, 0, 0)$ works. In fact, if $x = 0$, then (from the first eqn)

$$yz = 0 \qquad \Longrightarrow \qquad y = 0 \text{ or } z = 0.$$

If we pick $y = 0$, it will imply $z = 0$ and vice versa. Hence, if one var is 0, they all are (by symmetry of the 3 eqns). From the gradient, we find nonzero solns:

$$2x + yz = 0 \qquad \Longrightarrow \qquad x = -\tfrac{yz}{2}, \text{ and so}$$
$$2y + xz = 0 \qquad \Longrightarrow \qquad 2y = \tfrac{yz^2}{2} \qquad \Longrightarrow \qquad z = \pm 2.$$

Then $x = \mp y$, depending as $z = \pm 2$.

case (i) $z = 2$. Then $x = -y$ and the third eqn gives

$$4 - x^2 = 0 \implies x = \pm 2, y = \mp 2.$$

So we have crit pts $(2, -2, 2)$ and $(-2, 2, 2)$.

case (ii) $z = -2$. Then $x = y$ and the third eqn gives

$$-4 + x^2 = 0 \implies x = \pm 2, y = \pm 2.$$

So we have crit pts $(2, 2, -2)$ and $(-2, -2, -2)$.

The Hessian is

$$H_f(x, y, z) = \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix}$$

and only one crit pt is within the feasible set.

$$H_f(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ has eigenvals } 2, 2, 2.$$

So only $f(0,0,0) = 0$ is a (local and global) min.
Since the boundary is not included, we do not need to check it and we're done.

9. Use Lagrange multipliers to find the maximum value of $f(x, y) = 3x + 4y$ s.t. the constraint $x^2 + 4y^2 = 1$.
The constraint function is $g(x, y) = x^2 + 4y^2 - 1$ and the Lagrangian gives

$$\nabla f = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \lambda 2x \\ \lambda 8y \end{bmatrix} = \lambda \nabla g.$$

This gives

$$x = \tfrac{3}{2\lambda}, \quad y = \tfrac{4}{8\lambda} = \tfrac{1}{2\lambda},$$

which we put into the constraint to get

$$\begin{aligned} \left(\tfrac{3}{2\lambda}\right)^2 + 4\left(\tfrac{1}{2\lambda}\right)^2 &= 1 \\ \tfrac{9}{4\lambda^2} + \tfrac{4}{4\lambda^2} &= 1 \\ 13 &= 4\lambda^2 \\ \lambda &= \pm\tfrac{\sqrt{13}}{2} \end{aligned}$$

Thus,

$$x = \pm\tfrac{3}{\sqrt{13}}, \quad y = \pm\tfrac{1}{\sqrt{13}},$$

and the max and min are

$$f\left(\tfrac{3}{\sqrt{13}}, \tfrac{1}{\sqrt{13}}\right) = \tfrac{13}{\sqrt{13}} = \sqrt{13}, \quad \text{and} \quad f\left(-\tfrac{3}{\sqrt{13}}, -\tfrac{1}{\sqrt{13}}\right) = -\sqrt{13}.$$

10. Find the max and min of $f(x, y, z) = x - y - z$ s.t.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x + y + z &= 0. \end{aligned}$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$\begin{aligned} g_1(x, y, z) &= x^2 + y^2 + z^2 - 6 \\ g_2(x, y, z) &= x + y + z. \end{aligned}$$

$$\nabla f = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 2x + \lambda_2 \\ \lambda_1 2y + \lambda_2 \\ \lambda_1 2z + \lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

Working each eqn individually gives

$$x = \tfrac{1-\lambda_2}{2\lambda_1}, \quad y = \tfrac{-1-\lambda_2}{2\lambda_1}, \quad z = \tfrac{-1-\lambda_2}{2\lambda_1} = y,$$

so the second constraint gives

$$x + y + z = x + 2y = \tfrac{-1-3\lambda_2}{2\lambda_1} = 0 \implies \lambda_2 = -\tfrac{1}{3},$$

so

$$x = \tfrac{4/3}{2\lambda_1} = \tfrac{2}{3\lambda_1}, \quad y = z = \tfrac{-2/3}{2\lambda_1} = -\tfrac{1}{3\lambda_1}.$$

Putting into the first constraint,

$$\begin{aligned} \left(\tfrac{2}{3\lambda_1}\right)^2 + 2\left(\tfrac{1}{3\lambda_1}\right)^2 &= 6 \\ \tfrac{4}{9\lambda_1^2} + \tfrac{2}{9\lambda_1^2} &= 6 \\ \lambda_1^2 &= \tfrac{1}{9} \\ \lambda &= \pm\tfrac{1}{3}. \end{aligned}$$

Substituting into the eqs for x, y, z:

$$x = \tfrac{2}{3\lambda_1} = \pm 2, \quad y = z = -\tfrac{1}{3\lambda_1} = \mp 1.$$

Thus the max and min are

$$f(2, -1, -1) = 4 \quad \text{and} \quad f(-2, 1, 1) = -4.$$

11. The planes $x + y - z - 2w = 1$ and $x - y + z + 2w = 2$ intersect in a set $F \subseteq \mathbb{R}^4$. Find the point in F that is nearest to the origin.
Minimize (distance)^2 to the origin to minimize distance to the origin; i.e., minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ s.t.

$$\begin{aligned} x + y - z - 2w &= 1 \\ x - y + z + 2w &= 2 \end{aligned}$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$\begin{aligned} g_1(x, y, z, w) &= x + y - z - 2w - 1 \\ g_2(x, y, z, w) &= x - y + z + 2w - 2. \end{aligned}$$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \\ 2w \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \\ -\lambda_1 + \lambda_2 \\ -2\lambda_1 + 2\lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

The first two together give

$$x + y = \lambda_1, \quad x - y = \lambda_2,$$

and the second two give

$$z + 2w = -\tfrac{5}{2}(\lambda_1 - \lambda_2).$$

Putting these into the constraints (with $g_1 = 0, g_2 = 0$),

$$\begin{aligned} \lambda_1 + \tfrac{5}{2}(\lambda_1 - \lambda_2) &= 1 \\ \lambda_2 - \tfrac{5}{2}(\lambda_1 - \lambda_2) &= 2. \end{aligned}$$

Solving gives $\lambda_1 = \frac{17}{12}, \lambda_2 = \frac{19}{12}$. Then $x = \frac{3}{2}, y = -\frac{1}{12}$, so we obtain the new constraint $z + 2w = \frac{5}{12}$. This line is the set F, and it allows us to reduce f to a function of one variable:

$$\begin{aligned} f(x, y, z, w) &= f\left(\tfrac{3}{2}, -\tfrac{1}{12}, \tfrac{5}{12} - 2w, w\right) \\ &= f(w) \\ &= \left(\tfrac{3}{2}\right)^2 + \left(\tfrac{1}{12}\right)^2 + \left(\tfrac{5}{12} - 2w\right)^2 + w^2 \\ &= 5w^2 + -\tfrac{5}{3}w + \tfrac{175}{72}, \end{aligned}$$

with crit pt:

$$\begin{aligned} f'(w) = 10w - \tfrac{5}{3} &= 0 \\ w &= \tfrac{1}{6}. \end{aligned}$$

Thus, the closest point of F to the origin is $\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right)$. Since $f\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right) = \frac{55}{24}$, it is a distance of $\sqrt{\frac{55}{24}}$ away.

12. Use an appropriate minimum problem to show that

$$\sqrt[3]{abc} \le \frac{a + b + c}{3}, \qquad a, b, c > 0.$$

Define $f(x, y, z) = \sqrt[3]{xyz} - \frac{x+y+z}{3}$. We must show $\max_{x,y,z>0} f(x, y, z) \le 0$. So we are in the first quadrant. Crit pts:

$$\nabla f = \begin{bmatrix} \frac{(yz)^{1/3}}{3x^{1/3}} \\ \frac{(xz)^{1/3}}{3y^{1/3}} \\ \frac{(xy)^{1/3}}{3z^{1/3}} \end{bmatrix} \qquad \text{cannot be 0 for } x, y, z > 0 \text{ (or otherwise).}$$

So look to boundary (for the moment, allow $x, y, z \ge 0$). For $z = 0$, we are in the xy-plane (the "floor") and

$$f(x, y, z) = -\tfrac{(x+y)}{3} \le 0 \quad \text{since } x, y \ge 0.$$

This will only become larger when x, y get closer to 0, so

$$\max_{z=0} f(x,y,z) = f(0,0,0) = 0.$$

The situation is similar/symmetric for the other two planes. Therefore,

$$\max_{x,y,z>0} f(x,y,z) < 0.$$

13. Find the minimum and maximum value of $f(x,y,z) = x + 2y + 3z$. in the domain $D = \{3x^2 + 2y^2 + z^2 \leq 1\}$.
Constraint function is $g(x,y,z) = 3x^2 + 2y^2 + z^2 - 1$.

case (i) $g < 0$. Then $\lambda = 0$ and $\nabla f = 0$, so check crit pts.

$$\begin{aligned} 1 &= 0 \\ 2 &= 0 \quad \Longrightarrow ?? \\ 3 &= 0 \end{aligned}$$

The partials of a linear function cannot simultaneously vanish unless it is a constant function! So we have no critical points; the extrema of a linear function always occur on the boundary.

case (ii) $g = 0$. Then $\lambda \neq 0$ and we check the Lagrangian.

$$\begin{aligned} 1 &= -\lambda 6x \\ 2 &= -\lambda 4y \\ 3 &= -\lambda 2z \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= -\tfrac{1}{6\lambda} \\ y &= -\tfrac{1}{4\lambda} \\ z &= -\tfrac{1}{2\lambda} \end{aligned}$$

Then use the constraint

$$\begin{aligned} \tfrac{3}{36\lambda^2} + \tfrac{2}{16\lambda^2} + \tfrac{1}{4\lambda^2} &= 1 \\ 11 &= 24\lambda^2 \\ \lambda &= \pm\sqrt{\tfrac{11}{24}} \\ \tfrac{1}{\lambda} &= \pm 2\sqrt{\tfrac{6}{11}} \end{aligned}$$

This gives $x = \mp\frac{1}{3}\sqrt{\frac{6}{11}}, y = \mp\frac{1}{2}\sqrt{\frac{6}{11}}, z = \mp\sqrt{\frac{6}{11}}$, so

$$\begin{aligned} f(x_+, y_+, z_+) &= 13\sqrt{\tfrac{2}{33}} \quad \text{and} \\ f(x_-, y_-, z_-) &= -13\sqrt{\tfrac{2}{33}} \end{aligned}$$

are the max and min.

14. Find the local extreme values of $f(x, y) = x(1-y)$ s.t. the constraint $x^2 + 4y^2 \leq 1$.

case (i) $g < 0$. Then $\lambda = 0$ and $\nabla f = 0$, so check crit pts.

$$\begin{aligned} 1-y &= 0 \\ -x &= 0 \end{aligned} \implies \begin{aligned} y &= 1 \\ x &= 0 \end{aligned}$$

and $f(0, 1) = 0$.

case (ii) $g = 0$. Then $\lambda \neq 0$ and we check the Lagrangian.

$$\begin{aligned} 1 - y &= -\lambda 2x \\ -x &= -\lambda 8y \\ x^2 + 4y^2 - 1 &= 0. \end{aligned}$$

Then $x = \lambda 8y$, so

$$1 - y = \lambda^2 16y \implies y = \tfrac{1}{1-16\lambda^2}, \; x = \tfrac{8\lambda}{1-16\lambda^2}.$$

Plugging into constraint,

$$\begin{aligned} \tfrac{64\lambda^2}{(1-16\lambda^2)^2} + \tfrac{4}{(1-16\lambda^2)^2} &= 1 \\ 64\lambda^2 + 4 &= 1 - 32\lambda^2 + 256\lambda^4 \\ 256\lambda^4 - 96\lambda^2 - 3 &= 0 \\ \lambda^2 &= \frac{96 \pm \sqrt{12288}}{512} = \frac{3 \pm 2\sqrt{3}}{16} \\ \lambda^2 &= \frac{3 + 2\sqrt{3}}{16} \qquad (\text{since } \lambda^2 \geq 0) \\ \lambda &= \pm \frac{\sqrt{3 + 2\sqrt{3}}}{4}. \end{aligned}$$

Back into the formulae for x, y:

$$\begin{aligned} x &= \frac{\pm 2\sqrt{3 + 2\sqrt{3}}}{1 - (3 + 2\sqrt{3})} = \mp \frac{\sqrt{3 + 2\sqrt{3}}}{1 + \sqrt{3}} \\ y &= -\frac{1}{2 + 2\sqrt{3}} \end{aligned}$$

Then the max and min are $f(x_1, y) \approx 1.101$ and $f(x_2, y) \approx -1.101$.

15. (a) Max $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y &\leq 1 \\ x + y &\geq 1 \end{aligned}$$

$$\begin{aligned} -x + y &\geq 0 \\ 2x + y &\geq 1 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad &= 1 \\ -x - y \quad + v \qquad &= -1 \\ x - y \qquad + w \quad &= 0 \\ -2x - y \qquad\qquad + z &= -1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad &= 1 \\ x + y \quad - v \qquad &= 1 \\ x - y \qquad + w \quad &= 0 \\ 2x + y \qquad\qquad - z &= 1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(\bar{x}) = (3, -4, 0, 0, 0, 0) \cdot \bar{x}$ s.t.

$$\begin{bmatrix} 4 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \bar{x} &\geq 0 \\ u, v, w, z &\geq 0. \end{aligned}$$

(b) Minimize $f(x, y, z) = x - y + z$ s.t.

$$\begin{aligned} 4x + y - z &\geq 1 \\ 4x + y - z &\leq 1 \\ x + y + z &= 1 \\ x, y, z &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x, y) = -x + y - z$ s.t.

$$\begin{aligned} 4x + y - z &= 1 \\ x + y + z &= 1 \\ x, y, z &\geq 0, \end{aligned}$$

i.e.,

$$\begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \bar{\mathrm{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\bar{\mathrm{x}} \geq 0.$$

16. Find the max and min of $f(x, y) = 7x + 8y$ under the constraints

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 12$$
$$3x + y \leq 24.$$

The feasible region has vertices $(0, 0), (0, 6), (\frac{36}{5}, \frac{12}{5}), (8, 0)$.

$$f(0, 0) = 0$$
$$f(0, 6) = 48$$
$$f\left(\tfrac{36}{5}, \tfrac{12}{5}\right) = \tfrac{348}{5} = 69\tfrac{3}{5}$$
$$f(8, 0) = 56$$

So the max is at $(\frac{36}{5}, \frac{12}{5})$ and the min is at $(0, 0)$.
Standard form: maximize $f(x, y) = 7x + 8y$ s.t.

$$x + 2y + u \quad = 12$$
$$3x + y \quad + v = 24$$
$$x \geq 0$$
$$y \geq 0$$
$$u \geq 0$$
$$v \geq 0,$$

so maximize $f(\bar{\mathrm{x}}) = (7, 8, 0, 0) \cdot \bar{\mathrm{x}}$ s.t.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \bar{\mathrm{x}} = \begin{bmatrix} 12 \\ 24 \end{bmatrix}$$
$$\bar{\mathrm{x}} \geq 0$$
$$u, v \geq 0.$$