Classic Optimization Theory and Application

Homwork 1

13. Lagrange Multipliers

1. Use Lagrange multipliers to find the maximum value of $f(x,y) = 3x + 4y$ s.t. the constraint $x^2 + 4y^2 = 1$.
The constraint function is $g(x,y) = x^2 + 4y^2 - 1$ and the Lagrangian gives

$$\nabla f = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \lambda 2x \\ \lambda 8y \end{bmatrix} = \lambda \nabla g.$$

This gives

$$x = \tfrac{3}{2\lambda}, \quad y = \tfrac{4}{8\lambda} = \tfrac{1}{2\lambda},$$

which we put into the constraint to get

$$\begin{aligned} \left(\tfrac{3}{2\lambda}\right)^2 + 4\left(\tfrac{1}{2\lambda}\right)^2 &= 1 \\ \tfrac{9}{4\lambda^2} + \tfrac{4}{4\lambda^2} &= 1 \\ 13 &= 4\lambda^2 \\ \lambda &= \pm\tfrac{\sqrt{13}}{2} \end{aligned}$$

Thus,

$$x = \pm\tfrac{3}{\sqrt{13}}, \quad y = \pm\tfrac{1}{\sqrt{13}},$$

and the max and min are

$$f\left(\tfrac{3}{\sqrt{13}}, \tfrac{1}{\sqrt{13}}\right) = \tfrac{13}{\sqrt{13}} = \sqrt{13}, \quad \text{and} \quad f\left(-\tfrac{3}{\sqrt{13}}, -\tfrac{1}{\sqrt{13}}\right) = -\sqrt{13}.$$

2. Find the max and min of $f(x,y,z) = x - y - z$ s.t.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x + y + z &= 0. \end{aligned}$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$g_1(x,y,z) = x^2 + y^2 + z^2 - 6$$

$$g_2(x, y, z) = x + y + z.$$

$$\nabla f = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 2x + \lambda_2 \\ \lambda_1 2y + \lambda_2 \\ \lambda_1 2z + \lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

Working each eqn individually gives

$$x = \tfrac{1-\lambda_2}{2\lambda_1}, \quad y = \tfrac{-1-\lambda_2}{2\lambda_1}, \quad z = \tfrac{-1-\lambda_2}{2\lambda_1} = y,$$

so the second constraint gives

$$x + y + z = x + 2y = \tfrac{-1-3\lambda_2}{2\lambda_1} = 0 \implies \lambda_2 = -\tfrac{1}{3},$$

so

$$x = \tfrac{4/3}{2\lambda_1} = \tfrac{2}{3\lambda_1}, \quad y = z = \tfrac{-2/3}{2\lambda_1} = -\tfrac{1}{3\lambda_1}.$$

Putting into the first constraint,

$$\begin{aligned} \left(\tfrac{2}{3\lambda_1}\right)^2 + 2\left(\tfrac{1}{3\lambda_1}\right)^2 &= 6 \\ \tfrac{4}{9\lambda_1^2} + \tfrac{2}{9\lambda_1^2} &= 6 \\ \lambda_1^2 &= \tfrac{1}{9} \\ \lambda &= \pm\tfrac{1}{3}. \end{aligned}$$

Substituting into the eqs for x, y, z:

$$x = \tfrac{2}{3\lambda_1} = \pm 2, \quad y = z = -\tfrac{1}{3\lambda_1} = \mp 1.$$

Thus the max and min are

$$f(2, -1, -1) = 4 \quad \text{and} \quad f(-2, 1, 1) = -4.$$

14. Lagrange Multipliers with Inequalities

1. The planes $x + y - z - 2w = 1$ and $x - y + z + 2w = 2$ intersect in a set $F \subseteq \mathbb{R}^4$. Find the point in F that is nearest to the origin.
   Minimize (distance)^2 to the origin to minimize distance to the origin; i.e., minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ s.t.

$$x + y - z - 2w = 1$$
$$x - y + z + 2w = 2$$

Use the Lagrangian $\nabla f + \lambda \nabla g = 0$ with constraint functions

$$g_1(x, y, z, w) = x + y - z - 2w - 1$$
$$g_2(x, y, z, w) = x - y + z + 2w - 2.$$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \\ 2w \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \\ -\lambda_1 + \lambda_2 \\ -2\lambda_1 + 2\lambda_2 \end{bmatrix} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

The first two together give

$$x + y = \lambda_1, \quad x - y = \lambda_2,$$

and the second two give

$$z + 2w = -\tfrac{5}{2}(\lambda_1 - \lambda_2).$$

Putting these into the constraints (with $g_1 = 0, g_2 = 0$),

$$\lambda_1 + \tfrac{5}{2}(\lambda_1 - \lambda_2) = 1$$
$$\lambda_2 - \tfrac{5}{2}(\lambda_1 - \lambda_2) = 2.$$

Solving gives $\lambda_1 = \frac{17}{12}, \lambda_2 = \frac{19}{12}$. Then $x = \frac{3}{2}, y = -\frac{1}{12}$, so we obtain the new constraint $z + 2w = \frac{5}{12}$. This line is the set F, and it allows us to reduce f to a function of one variable:

$$\begin{aligned} f(x, y, z, w) &= f\left(\tfrac{3}{2}, -\tfrac{1}{12}, \tfrac{5}{12} - 2w, w\right) \\ &= f(w) \\ &= \left(\tfrac{3}{2}\right)^2 + \left(\tfrac{1}{12}\right)^2 + \left(\tfrac{5}{12} - 2w\right)^2 + w^2 \\ &= 5w^2 + -\tfrac{5}{3}w + \tfrac{175}{72}, \end{aligned}$$

with crit pt:

$$\begin{aligned} f'(w) &= 10w - \tfrac{5}{3} = 0 \\ w &= \tfrac{1}{6}. \end{aligned}$$

Thus, the closest point of F to the origin is $\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right)$.
Since $f\left(\frac{3}{2}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{6}\right) = \frac{55}{24}$, it is a distance of $\sqrt{\frac{55}{24}}$ away.

2. Use an appropriate minimum problem to show that

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}, \qquad a, b, c > 0.$$

Define $f(x, y, z) = \sqrt[3]{xyz} - \frac{x+y+z}{3}$. We must show $\max_{x,y,z>0} f(x, y, z) \leq 0$. So we are in the first quadrant. Crit pts:

$$\nabla f = \begin{bmatrix} \frac{(yz)^{1/3}}{3x^{1/3}} \\ \frac{(xz)^{1/3}}{3y^{1/3}} \\ \frac{(xy)^{1/3}}{3z^{1/3}} \end{bmatrix} \qquad \text{cannot be 0 for } x, y, z > 0 \text{ (or otherwise).}$$

So look to boundary (for the moment, allow $x, y, z \geq 0$). For $z = 0$, we are in the xy-plane (the "floor") and

$$f(x, y, z) = -\frac{(x+y)}{3} \leq 0 \quad \text{since } x, y \geq 0.$$

This will only become larger when x, y get closer to 0, so

$$\max_{z=0} f(x, y, z) = f(0, 0, 0) = 0.$$

The situation is similar/symmetric for the other two planes. Therefore,

$$\max_{x,y,z>0} f(x, y, z) < 0.$$

3. Use an appropriate minimum problem to show that

$$(a_1 a_2 \ldots a_n)^{1/n} \leq \tfrac{1}{n}(a_1 a_2 \ldots a_n), \qquad a_i > 0.$$

Same soln as previous problem.

4. Find the minimum and maximum value of $f(x, y, z) = x + 2y + 3z$. in the domain $D = \{3x^2 + 2y^2 + z^2 \leq 1\}$.
Constraint function is $g(x, y, z) = 3x^2 + 2y^2 + z^2 - 1$.

case (i) $g < 0$. Then $\lambda = 0$ and $\nabla f = 0$, so check crit pts.

$$\begin{aligned} 1 &= 0 \\ 2 &= 0 \quad \Longrightarrow ?? \\ 3 &= 0 \end{aligned}$$

The partials of a linear function cannot simultaneously vanish unless it is a constant function! So we have no critical points; the extrema of a linear function always occur on the boundary.

case (ii) $g = 0$. Then $\lambda \neq 0$ and we check the Lagrangian.

$$\begin{aligned} 1 &= -\lambda 6x \\ 2 &= -\lambda 4y \\ 3 &= -\lambda 2z \end{aligned} \implies \begin{aligned} x &= -\tfrac{1}{6\lambda} \\ y &= -\tfrac{1}{4\lambda} \\ z &= -\tfrac{1}{2\lambda} \end{aligned}$$

Then use the constraint

$$\begin{aligned} \tfrac{3}{36\lambda^2} + \tfrac{2}{16\lambda^2} + \tfrac{1}{4\lambda^2} &= 1 \\ 11 &= 24\lambda^2 \\ \lambda &= \pm\sqrt{\tfrac{11}{24}} \\ \tfrac{1}{\lambda} &= \pm 2\sqrt{\tfrac{6}{11}} \end{aligned}$$

This gives $x = \mp\frac{1}{3}\sqrt{\frac{6}{11}}, y = \mp\frac{1}{2}\sqrt{\frac{6}{11}}, z = \mp\sqrt{\frac{6}{11}}$, so

$$f(x_+, y_+, z_+) = 13\sqrt{\tfrac{2}{33}} \quad \text{and}$$

$$f(x_-, y_-, z_-) = -13\sqrt{\tfrac{2}{33}}$$

are the max and min.

15. Lagrange Multipliers with Inequalities

1. Find the local extreme values of $f(x,y) = x(1-y)$ s.t. the constraint $x^2 + 4y^2 \leq 1$.

case (i) $g < 0$. Then $\lambda = 0$ and $\nabla f = 0$, so check crit pts.

$$\begin{aligned} 1-y &= 0 \\ -x &= 0 \end{aligned} \implies \begin{aligned} y &= 1 \\ x &= 0 \end{aligned}$$

and $f(0,1) = 0$.

case (ii) $g = 0$. Then $\lambda \neq 0$ and we check the Lagrangian.

$$\begin{aligned} 1-y &= -\lambda 2x \\ -x &= -\lambda 8y \\ x^2 + 4y^2 - 1 &= 0. \end{aligned}$$

Then $x = \lambda 8y$, so

$$1 - y = \lambda^2 16y \implies y = \tfrac{1}{1-16\lambda^2},\ x = \tfrac{8\lambda}{1-16\lambda^2}.$$

Plugging into constraint,

$$\begin{aligned} \tfrac{64\lambda^2}{(1-16\lambda^2)^2} + \tfrac{4}{(1-16\lambda^2)^2} &= 1 \\ 64\lambda^2 + 4 &= 1 - 32\lambda^2 + 256\lambda^4 \\ 256\lambda^4 - 96\lambda^2 - 3 &= 0 \\ \lambda^2 &= \frac{96 \pm \sqrt{12288}}{512} = \frac{3 \pm 2\sqrt{3}}{16} \\ \lambda^2 &= \frac{3 + 2\sqrt{3}}{16} \qquad \text{(since } \lambda^2 \geq 0) \\ \lambda &= \pm\frac{\sqrt{3+2\sqrt{3}}}{4}. \end{aligned}$$

Back into the formulae for x, y:

$$\begin{aligned} x &= \frac{\pm 2\sqrt{3+2\sqrt{3}}}{1-(3+2\sqrt{3})} = \mp\frac{\sqrt{3+2\sqrt{3}}}{1+\sqrt{3}} \\ y &= -\frac{1}{2+2\sqrt{3}} \end{aligned}$$

Then the max and min are $f(x_1, y) \approx 1.101$ and $f(x_2, y) \approx -1.101$.

2. Find the global max and min of

$$f(x,y) = \sqrt{x^2+y^2}\log\left(\sqrt{x^2+y^2}\right)$$

s.t. the constraint

$$\sqrt{x^2+y^2} \le 1.$$

In polar coordinates, this becomes a much simpler problem:

$$\text{maximize} \quad f(r) = r\log r, \qquad \text{s.t. } 0 \le r \le 1.$$

Using Math 09A:

$$\begin{aligned} f'(r) = 1 + \log r &= 0 \\ \log r &= -1 \\ r &= e^{-1} = \tfrac{1}{e}. \end{aligned}$$

Also, since $f'(0) < 0$, there is a local max when $r = 0$; and since $f'(1) > 0$, there is a local max when $r = 1$. How to interpret this? The graph of f is apparently a surface of revolution swept out by a convex arc over the unit interval which goes from $f(0) = \lim_{r\to 0} f(r) = 0$ down to a min at $f(1/e) = -\frac{1}{e}$, and then up to a max at $f(1) = 0$.

So there is an isolated local/global max $f(0) = 0$, an entire circle of local/global mins

$$\{(x,y) : x^2 + y^2 = e^{-2}\},$$

and an entire circle of local/global maxs

$$\{(x,y) : x^2 + y^2 = 1\}.$$

16. Linear Programming

1. Find the max and min of $f(x, y) = 7x + 8y$ under the constraints

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 12 \\ 3x + y &\leq 24. \end{aligned}$$

The feasible region has vertices $(0,0), (0,6), (\frac{36}{5}, \frac{12}{5}), (8,0)$.

$$\begin{aligned} f(0,0) &= 0 \\ f(0,6) &= 48 \\ f\left(\tfrac{36}{5}, \tfrac{12}{5}\right) &= \tfrac{348}{5} = 69\tfrac{3}{5} \\ f(8,0) &= 56 \end{aligned}$$

So the max is at $(\frac{36}{5}, \frac{12}{5})$ and the min is at $(0,0)$.
Standard form: maximize $f(x, y) = 7x + 8y$ s.t.

$$\begin{aligned} x + 2y + u \quad &= 12 \\ 3x + y \quad + v &= 24 \\ x &\geq 0 \\ y &\geq 0 \\ u &\geq 0 \\ v &\geq 0, \end{aligned}$$

so maximize $f(\bar{\mathrm{x}}) = (7, 8, 0, 0) \cdot \bar{\mathrm{x}}$ s.t.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \bar{\mathrm{x}} = \begin{bmatrix} 12 \\ 24 \end{bmatrix}$$

$$\begin{aligned} \bar{\mathrm{x}} &\geq 0 \\ u, v &\geq 0. \end{aligned}$$

17. Linear Programming

1. A tailor produces two types of gowns for the ladies of the town. There are the frilly ones, using 10 yards of lace and 4 yard of fabric, and the more plain gowns, using 1 yard of lace and 8 yards of fabric. Times are tight, and he only makes money by selling the lace and the fabric - labor and the rest of the materials are to be delivered free of charge. He can buy 300 yards of lace and 300 yards of fabric each season, and his seamstresses can sew together no more than 50 gowns per season. If he can earn 5 ducats per yard of lace and 2 ducats per yard of fabric, how many gowns of each type should he plan on making per season?

   Let x denote the number of frilly gowns and y the number of plain ones. The profit function is

$$f(x,y) = (10 \cdot 5 + 4 \cdot 2)x + (1 \cdot 5 + 8 \cdot 2)y = 58x + 21y$$

   and the constraints are

$$\begin{aligned} 10x + y &\leq 300 && \text{(lace)} \\ 4x + 8y &\leq 300 && \text{(fabric)} \\ x, y &\geq 0. \end{aligned}$$

   The vertices are $(0,0)$, $\left(0, \frac{75}{2}\right)$, $(30,0)$, $\left(\frac{525}{19}, \frac{450}{19}\right)$. Checking the vertices,

$$\begin{aligned} f(0,0) &= 0 \\ f\left(0, \tfrac{75}{2}\right) &= \tfrac{1575}{2} \\ f(30,0) &= 1740 \\ f\left(\tfrac{525}{19}, \tfrac{450}{19}\right) &= 2100. \end{aligned}$$

   So he should make 27 $\left(= \lfloor \frac{525}{19} \rfloor\right)$ frilly dresses and 23 $\left(= \lfloor \frac{450}{19} \rfloor\right)$ plain dresses, and maybe a couple purses or fancy hats.

18. Linear Programming - Standard Form

1. (a) Max $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y &\leq 1 \\ x + y &\geq 1 \\ -x + y &\geq 0 \\ 2x + y &\geq 1 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad\qquad &= 1 \\ -x - y \quad + v \qquad\qquad &= -1 \\ x - y \qquad\quad + w \qquad &= 0 \\ -2x - y \qquad\qquad\quad + z &= -1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(x, y) = 3x - 4y$ s.t.

$$\begin{aligned} 4x - y + u \qquad\qquad\qquad &= 1 \\ x + y \quad - v \qquad\qquad &= 1 \\ x - y \qquad\quad + w \qquad &= 0 \\ 2x + y \qquad\qquad\quad - z &= 1 \\ x, y, u, v, w, z &\geq 0, \end{aligned}$$

i.e., maximize $f(\bar{x}) = (3, -4, 0, 0, 0, 0) \cdot \bar{x}$ s.t.

$$\begin{bmatrix} 4 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\bar{x} \geq 0$$
$$u, v, w, z \geq 0.$$

(b) Minimize $f(x, y, z) = x - y + z$ s.t.

$$\begin{aligned} 4x + y - z &\geq 1 \\ 4x + y - z &\leq 1 \\ x + y + z &= 1 \\ x, y, z &\geq 0. \end{aligned}$$

So this becomes: maximize $f(x, y) = -x + y - z$ s.t.

$$4x + y - z = 1$$
$$x + y + z = 1$$
$$x, y, z \geq 0,$$

i.e.,

$$\begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \bar{\mathrm{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\bar{\mathrm{x}} \geq 0.$$