Exercise Solutions to Functional Analysis

Note: References refer to M. Schechter, "Principles of Functional Analysis"

Exersize 1. Let $\phi_1, \ldots, \phi_n$ be an orthonormal set in a Hilbert space H. Show that

$$\left\| f - \sum_{k=1}^{n} \alpha_k \phi_k \right\| \geq \left\| f - \sum_{k=1}^{n} (f, \phi_k)\phi_k \right\|$$

Solution 1. Since $f \in H$, we know that $f \in V \subset H$, where V is some finite dimensional subspace of H, say $\dim V = m \geq n$. We may assume that V contains $\phi_1, \phi_2, \ldots, \phi_n$. Let $\{\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m\}$ be an orthonormal basis for V (this is possible by extending $\{\phi_1, \ldots, \phi_n\}$ to an orthonormal basis via Gram-Schmidt).

Now write $f = \sum_k^m \beta_k \phi_k$. Then we have

$$f - \sum_k^n (f, \phi_k)\phi_k = \sum_{k=n+1}^{m} \beta_k \phi_k$$

Hence

$$\left\| f - \sum_k^n (f, \phi_k)\phi_k \right\|^2 = \sum_{k=n+1}^{m} \beta_k^2$$

since $\|\phi_k\| = 1$ for $1 \leq k \leq m$. On the other hand,

$$f - \sum_k^n \alpha_k \phi_k = \sum_k^n (\beta_k - \alpha_k)\phi_k + \sum_{k=n+1}^{m} \beta_k \phi_k$$

Hence

$$\left\| f - \sum_k^n \alpha_k \phi_k \right\|^2 = \sum_k^n (\beta_k - \alpha_k)^2 + \sum_{k=n+1}^{m} \beta_k^2$$

It is clear now that

$$\left\| f - \sum_k^n \alpha_k \phi_k \right\|^2 \geq \left\| f - \sum_k^n (f, \phi_k)\phi_k \right\|^2$$

It follows that

$$\left\| f - \sum_k^n \alpha_k \phi_k \right\| \geq \left\| f - \sum_k^n (f, \phi_k)\phi_k \right\|$$

We're done. □

Exersize 2. Let c denote the set of all elements $(\alpha_1, \alpha_2, \dots) \in l_\infty$ such that $\{\alpha_n\}$ is a convergent sequence, and let c_0 be the set of all such elements for which $\alpha_n \to 0$ as $n \to \infty$. Show that c and c_0 are Banach spaces.

Solution 2. It is known that l_∞ is a Banach space. The linear structure, as well as the norm, of c and c_0 is inherited from that of l_∞. To show completeness of c and c_0, it will suffice to show that c and c_0 are closed in l_∞. We show that c is closed in l_∞ and c_0 is closed in c.

Now, suppose that $\{x_k\}$ is a sequence in c converging to some $x \in l_\infty$. Then, for any given $\epsilon > 0$, we know that there exists k_0 such that

$$\sup_j |x^j_{k_0} - x^j| < \frac{\epsilon}{3}$$

Hence for all j we have

$$|x^j_{k_0} - x^j| < \frac{\epsilon}{3}$$

We also know that, since $\left\{x^j_{k_0}\right\}_{j\in\mathbb{N}}$ is a convergent sequence, it is Cauchy, so there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$, we have

$$|x^n_{k_0} - x^m_{k_0}| < \frac{\epsilon}{3}$$

From this we deduce:

$$|x^n - x^m| \leq |x^n - x^n_{k_0}| + |x^n_{k_0} - x^m_{k_0}| + |x^m_{k_0} - x^m| < \epsilon$$

for all $n \geq m \geq N$. This shows that $\{x\}$ is Cauchy, hence convergent. Thus $x \in c$. This shows that c contains all its limit points, hence closed. So c, being a closed subset of a complete space, must itself be complete.

Similarly, let us suppose that $\{x_k\}$ is a sequence in c_0 that converges to $x \in l_\infty$. Then for all $\epsilon > 0$, there exists k_0 such that

$$\sup_j |x^j_{k_0} - x^j| < \frac{\epsilon}{2}$$

Hence for all j we have

$$|x^j_{k_0} - x^j| < \frac{\epsilon}{2}$$

On the other hand, since $x_{k_0} \in c_0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|x^n_{k_0}| < \frac{\epsilon}{2}$$

Hence for all $n \geq N$ we have

$$|x^n| \leq |x^n_{k_0}| + |x^n_{k_0} - x^n| < \epsilon$$

So $x = \left\{x^j\right\}$ converges to 0, hence $x \in c_0$. So c_0 is complete. □

Exersize 3. Show that the norm of an element is never negative.

Solution 3. Suppose that $\|x\| < 0$. But then $0 = \|x - x\| \leq \|x\| + |-1|\,\|x\| < 0$. Can't be! □

Exersize 4. If x, y are are elements of a Hilbert space H and satisfy $\|x + y\| = \|x\| + \|y\|$, show that either $x = cy$ or $y = cx$ for some scalar $c \geq 0$.

Solution 4. Well, if either $x = 0$ or $y = 0$, the result holds vacuously. So assume that neither x nor y is equal to zero. We may assume that $x, y \in V \subset H$, with $\dim V = m < \infty$. If $m = 1$, we're done. If not, we may take for a basis of V the set $\{x_1, x_2, \ldots, x_m\}$ of orthonormal vectors where $x_1 = x/\|x_1\|$. This can be done via Gram-Schmidt. Write $y = \alpha_1 x_1 + \cdots + \alpha_m x_m$. Then we have

$$\|x_1 + (\alpha_1 x_1 + \cdots + \alpha_m x_m)\| = \|x_1\| + \|\alpha_1 x_1 + \cdots + \alpha_m x_m\|$$

From this we immediately deduce

$$2|(x_1, \alpha_1 x_1 + \cdots + \alpha_m x_m)| = 2\|x_1\| \|\alpha_1 x_1 + \cdots + \alpha_m x_m\|$$

By orthonormality of $\{x_1, \ldots, x_m\}$, we get from above:

$$|\alpha_1| = \sqrt{\alpha_1^2 + \cdots + \alpha_m^2}$$

From this we immediately deduce that

$$\alpha_2 = \cdots = \alpha_m = 0$$

Hence we must have

$$y = \alpha_1 x_1 = \frac{\alpha_1}{\|x\|} x$$

We're done. □

Exersize 5. Show that in a normed vector space

$$\sum_{j=1}^{\infty} v_j = \sum_{j=1}^{n} v_j + \sum_{j=n+1}^{\infty} v_j$$

Solution 5. In a normed vector space the notion of the limit makes sense, and in fact all standard results regarding algebra of limits hold. In particular we have, by definition (for n fixed)

$$\sum_{j=1}^{\infty} v_j = \lim_{k\to\infty} \sum_{j=1}^{k} v_j = \lim_{k\to\infty, k\geq n} \sum_{j=1}^{k} v_j$$

On the other hand for $k \geq n$ we have

$$\sum_{j=1}^{k} v_j = \sum_{j=1}^{n} v_j + \sum_{j=n+1}^{k} v_j$$

Hence

$$\sum_{j=1}^{\infty} v_j = \lim_{k\to\infty} \left(\sum_{j=1}^{n} v_j + \sum_{j=n+1}^{k} v_j \right) = \sum_{j=1}^{n} v_j + \lim_{k\to\infty} \sum_{j=n+1}^{k} v_j = \sum_{j=1}^{n} v_j + \sum_{j=n+1}^{\infty} v_j$$

□

Exersize 6. For $f = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, show that

$$\|f\|_0 = \max_k |\alpha_k|$$

and

$$\|f\|_1 = \sum_{k=1}^{n} |\alpha_k|$$

are norms. Is $\mathbb{R}^n$ complete with respect to either of them?

Solution 6. It is easy to see that (11) and (12) of the definition of a norm on p. 7 hold trivially in both cases. So we only verify the triangle inequality.

Write $f = (\alpha_1, \ldots, \alpha_n)$ and $g = (\beta_1, \ldots, \beta_n)$. Then we have

$$\|f+g\|_0 = \max_k |\alpha_k + \beta_k| \leq \max_k(|\alpha_k| + |\beta_k|)$$

which follows from the triangle inequality of $|\cdot|$ on $\mathbb{R}$. On the other hand, it isn't hard to see that

$$\max_k(|\alpha_k| + |\beta_k|) \leq \max_k |\alpha_k| + \max_k |\beta_k| = \|f\|_0 + \|g\|_0$$

In the second case,

$$\|f+g\|_1 = \sum_{k=1}^{n} |\alpha_k + \beta_k| \leq \sum_{k=1}^{n}(|\alpha_k| + |\beta_k|) = \|f\|_1 + \|g\|_1$$

Now, we know that $\mathbb{R}^n$ is complete with respect to the standard Euclidean norm. But all norms on a finite dimensional vector space are equivalent. So $\mathbb{R}^n$ is complete with respect to the two norms above.

More directly, it isn't hard to see that:

$$\|f\|_0 \leq \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}$$

and

$$\|f\|_1 \leq \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}$$

□

Exersize 7. If F is a bounded linear functional on a normed vector space X, show that

$$\|F\| = \sup_{\|x\| \leq 1} |F(x)| = \sup_{\|x\|=1} |F(x)|$$

Solution 7. By definition we have

$$\|F\| = \sup_{x \neq 0} \frac{|F(x)|}{\|x\|}$$

But

$$\sup_{x \neq 0} \frac{|F(x)|}{\|x\|} = \sup_{x \neq 0} \left| F\left(\frac{x}{\|x\|}\right) \right| = \sup_{\|x\|=1} |F(x)|$$

On the other hand,

$$\sup_{\|x\|=1} |F(x)| = \sup_{x \neq 0} \frac{|F(x)|}{\|x\|} \geq \sup_{\|x\| \leq 1} \frac{|F(x)|}{\|x\|} \geq \sup_{\|x\| \leq 1} |F(x)| \geq \sup_{\|x\|=1} |F(x)|$$

This verifies the claim. □

Exersize 8. A functional $F(x)$ is called *additive*if $F(x+y) = F(x) + F(y)$. If F is additive, show that $F(\alpha x) = \alpha F(x)$ for all rational α.

Solution 8. Well, $F(0) = F(0+0) = F(0) + F(0)$. So $F(0) = 0$.

By definition, for any $n > 0$ an integer, we have

$$nx = \underbrace{x + x + \cdots + x}_{n \text{ - times}}$$

Hence we must have $F(nx) = nF(x)$.

Now, for $m > 0$, $F(x) = F(mx/m) = mF(x/m)$. So $F(x/m) = F(x)/m$. Now it immediately follows that for $n \geq 0, m > 0$, we have $F(nx/m) = nF(x)/m$.

Finally, $0 = F(x - x) = F(x + (-x)) = F(x) + F(-x)$. So $F(-x) = -F(x)$. This completes the proof. □

Exersize 9. Show that an additive functional is continuous everywhere if it is continuous at one point.

Solution 9. Suppose that F is an additive function which is continuous at some x_0. We shall show that F is bounded. That is, there exists $C > 0$ such that for all x we have

$$|F(x)| \leq C \|x\|$$

Since F is continuous at x_0, we know that, for arbitrary $\epsilon > 0$ fixed, there exists a $\delta > 0$ such that for all $\|x - x_0\| \leq \delta$ we have $|F(x) - F(x_0)| \leq \epsilon$. Now take $y \neq 0$. If $\|y\|$ is not rational, we can always pick $\|y\|/2 < M < \|y\|$ rational sufficiently close to $\|y\|$. Set

$$x = x_0 + \frac{\delta}{2M} y$$

Then it is clear that

$$\|x - x_0\| < \delta$$

Hence we have (also since F is additive):

$$|F(x - x_0)| \leq \epsilon$$

By the previous problem, since $\delta/2M$ is rational, we must have

$$\epsilon \geq |F(x - x_0)| = \left| F\left(\frac{\delta}{2M} y\right) \right| = \frac{\delta}{2M} |F(y)|$$

Hence

$$|F(y)| \leq \frac{\epsilon}{\delta} 2M < \frac{2\epsilon}{\delta} \|y\|$$

This construction works for all y, with $\frac{2\epsilon}{\delta}$ being independent of y. Hence F is bounded, thus continuous everywhere. □

Exersize 10. Show that the set I^ω is compact.

Solution 10. Since l_∞ is a metric space, we know that a subset of a metric space is compact if and only if it is sequentially compact. So we show that I^ω is sequentially compact: that is, every sequence has a convergent subsequence.

Let $x(i,j)_{i,j \in \mathbb{N}}$ be a sequence in l_∞ such that $|x(i,j)| \leq 1/j$ for all i, j. Clearly for each fixed j, $x(i,j)$ has a subsequence $x(i_k, j)$ that converges. Now let $y^1(i,j)$ be a subsequence of $x(i,j)$ such that $y^1(i,1) \to \tilde{y}^1$. Let $y^2(i,j)$ be a subsequence of $y^1(i,j)$, such that $y^2(i,2) \to \tilde{y}^2$. In general, let $y^n(i,j)$ be a subsequence of $y^{n-1}(i,j)$, such that $y^n(i,n) \to \tilde{y}^n$. Now let $z(i,j) = y^i(i,j)$. We claim that $z(i,j), (\tilde{y}^1, \tilde{y}^2, \dots) \in I^\omega$ and that $z(i,j) \to (\tilde{y}^1, \tilde{y}^2, \dots)$ as $i \to \infty$.

Clearly, by construction, for each fixed i, $|z(i,j)| \leq 1/j$, since it is a germ of the sequence $x(i,j)_{j \in \mathbb{N}}$. Since $\tilde{y}^j$ is the limit of a subsequence of $x(i,j)_{i \in \mathbb{N}}$, and $x(i,j) \leq 1/j$ for all i, we must have also $|\tilde{y}^j| \leq 1/j$. Hence $(\tilde{y}^j)_{j \in \mathbb{N}} \in I^\omega$.

Now suppose that $\epsilon > 0$ is fixed. Let N be so large that $1/N < \epsilon/2$. Let $M_1, M_2, \dots, M_N$ be such that for all $k \geq M_i$, $|y^i(k,i) - \tilde{y}^i| < \epsilon$. Let $M = \max\{\max_i \{M_i\}, N\}$. Then observe that for all $k \geq M$, $|z(k,j) - \tilde{y}^j| < \epsilon$ when

$j \leq N$, and $|z(k,j) - y^j| \leq |z(k,j)| + |y^j| \leq 1/N + 1/N = 2/N < \epsilon$ when $j > N$. Hence we have

$$\sup_j |z(k,j) - \tilde{y}^j| < \epsilon \text{ for all } k > M$$

So, as claimed, $z(i,j) \to (\tilde{y}^j)$ in l_∞-norm. This proves sequential compactness of I^ω, and hence compactness. □

Exersize 11. Let M be a totally bounded subset of a normed vector space X. Show that for each $\epsilon > 0$, M has a finite ϵ-net $N \subset M$.

Solution 11. We proceed by contradiction. Suppose that M is totally bounded, and for some fixed $\epsilon > 0$, there is no finite ϵ-net N contained entirely in M. Then given $x_1 \in M$, there exists $x_2 \in M$ such that $\|x_2 - x_1\| > \epsilon$. Similarly there must exist $x_3 \in M$ such that $\|x_i - x_3\| > \epsilon$, with $i = 1, 2$. In general, we can continue this process and obtain $x_n \in M$ such that $\|x_n - x_i\| > \epsilon$ with $i = 1, 2, \ldots, n-1$. Let $S = \{x_1, x_2, \ldots\}$. However, M is totally bounded, so there exists a finite $\epsilon/2$-net of M, say given by $\{w_1, w_2, \ldots, w_k\}$ such that the balls $B(w_i, \epsilon/2)$, with $i = 1, 2, \ldots, k$ cover M. Since there are infinitely many distinct terms in the set $S \subset M$, for some $i \neq j$, $x_i, x_j \in S$ and $x_i, x_j \in B(w_l, \epsilon/2)$ for some $1 \leq l \leq k$. But then $\|x_i - x_j\| \leq \epsilon/2$, which cannot be by construction of S. This is a contradiction. □

Exersize 12. Prove that if M is finite dimensional, then we can take $\theta = 1$ in Lemma 4.7

Solution 12. Let $\theta_n = (n-1)/n < 1$ for $n \geq 2$. We know then that there exists $x_n \in M^c$, such that $d(x_n, M) \geq \theta_n$ and $\|x_n\| = 1$ (in some fixed norm on X: X has many admissible norms, but all are equivalent). Since X is finite-dimensional and (x_n) is a bounded sequence, we know that x_n has a convergent subsequence, say $x_{n_k} \to x \in X$. But then $\|x\| = 1$ (this is obvious from the continuity of the norm, and the fact that $\|x_n\| = 1$ for all n) and $d(x, M) \geq (n_k - 1)/n_k$ for all n_k. Hence $d(x, M) \geq 1$. We're done. □

Exersize 13. Show that if X is infinite dimensional and K is one-to-one operator in $K(X)$, then $K - I$ cannot be in $K(X)$.

Solution 13. Let $S \subset X$ be the unit sphere (that is, the set of all vectors of unit norm). For convenience, let $Sp(x_1, x_2, \ldots, x_k)$ denote the space spanned by the vectors x_i, $i = 1, 2, \ldots, k$. Now, pick $x_1 \in S$. Clearly $Sp(x_1) \neq X$, so there exists $x_2 \in S$ such that $\|x_1 - x_2\| > 1/2$. Now, obviously $Sp(x_1, x_2) \neq X$. So there exists $x_3 \in S$ such that $\|x_1 - x_3\|, \|x_2 - x_3\| > 1/2$. Continue this process, at n^{th} step selecting $x_n \in S$ such that $\|x_i - x_n\| > 1/2$ with $i = 1, 2, \ldots, n-1$. This is possible since $Sp(x_1, x_2, \ldots, x_{n-1}) \neq X$ (since X is infinite dimensional). Now, the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded. Keep this in mind...

Now suppose that $A = K - I$ is compact. But then there exists a subsequence (x_{n_k}) such that Ax_{n_k} converges; so in particular is Cauchy. Let, for convenience, $y_k = x_{n_k}$. Thus, for $\epsilon < 1/2$ there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$ we have

$$\|Ay_n - Ay_m\| < \epsilon$$

Hence

$$\|y_m - y_n\| - \|Ky_m - Ky_n\| \leq \|(y_m - y_n) - (Ky_m - Ky_n)\| = \|(Ky_n - y_n) - (Ky_m - y_m)\| < \epsilon$$

But then

$$\|y_m - y_n\| < \epsilon + \|Ky_m - y_n\|$$

Now, K is compact, so there exists a subsequence $z_k = y_{n_k}$ such that Kz_k converges, so in particular is Cauchy. Hence for N large enough, $\epsilon + \|Kz_m - z_n\| < 1/2$ for all $n \geq m \geq N$. So

$$\|z_m - z_n\| < \epsilon + \|Kz_m - z_n\| < 1/2$$

which cannot be, since $\|z_m - z_n\| > 1/2$ by construction of the sequence (x_n), of which (z_n) is a subsequence.

So A cannot be compact. This completes the proof. □

Exersize 14. Let X be a vector space which can be made into a Banach space by introducing either of two different norms. Suppose that convergence with respect to the first norm always implies convergence with respect to the second norm. Show that the norms are equivalent.

Solution 14. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the two norms. It will suffice to show only that there exists $C > 0$ such that for all $x \in X$ nonzero one has

$$\|x\|_1 \leq C\|x\|_2$$

for then a similar argument will establish the existence of $D > 0$ such that

$$\|x\|_2 \leq D\|x\|_1$$

from which equivalence of the two norms could be deduced easily.

Suppose, for a contradiction, that for all $n \in N$, there exists $x_n \in X$ nonzero such that

$$\|x_n\|_1 > n\|x_n\|_2$$

But then we must have

$$\frac{\|x_n\|_2}{\|x_n\|_1} < \frac{1}{n}$$

Hence $x_n/\|x_n\|_1 \to 0$ in the $\|\cdot\|_2$-norm. But then, by hypothesis, $x_n/\|x_n\|_1 \to 0$ in the $\|\cdot\|_1$-norm. This cannot be, since $\|x_n/\|x_n\|_1\|_1 = 1$ for all n. This is a contradiction. We're done. □

Exersize 15. Suppose $A \in B(X, Y), K \in K(X, Y)$, where X, Y are Banach spaces. If $R(A) \subset R(K)$, show that $A \in K(X, Y)$.

Solution 15. Since both A and K are linear operators, we know that $R(A), R(K) \subset Y$ are vector subspaces of Y. Now, we also know that since K is compact, $R(K)$ is finite-dimensional (see next problem below). But then $A : X \to R(A) \subset R(K)$ is a bounded operator into a finite dimensional space. Hence A is compact. To see this: let (x_n) be a bounded sequence in X. Since A is bounded, $A(x_n)$ is a bounded sequence in $R(A)$. Since $R(A)$ is finite-dimensional, $A(x_n)$ must have a convergent subsequence. We're done. □

Exersize 16. Suppose X, Y are Banach spaces and $K \in K(X, Y)$. If $R(K) = Y$, show that Y is finite dimensional.

Solution 16. Let $\tilde{X} = X/N(K)$. Now, $N(K)$ is certainly closed in X. Define the norm on X/N

$$\|\cdot\|_{\tilde{X}} : \tilde{X} \to \mathbb{R} \text{ by } \|\tilde{x}\|_{\tilde{X}} = \inf_{z \in N(K)} \|z - \tilde{x}\|$$

where $\|\cdot\|$ is the norm on X. It isn't hard to verify that the norm above is well-defined, and actually *is* a norm.

Let $\tilde{K} : \tilde{X} \to Y$ be the homomorphism (of vector spaces) induced by the homomorphism K - that is, $\tilde{K}([x]) = K(x)$. Then this induced homomorphism is actually a vector space isomorphism (fundamental theorem of vector space homomorphisms - isn't hard to verify). But more is true - $\tilde{K}$ is compact! To see this, observe that since $N(K)$ is closed, for all $x \notin N(K)$, there exists $z \in N(K)$ such that $\|[x]\|_{\tilde{X}} = \|x - z\|$. Now, let us suppose that $[x_1], [x_2], \ldots$ is a sequence in $\tilde{X}$ that is bounded, say by $C > 0$. But then there exist $z_1, z_2, \cdots \in N(K)$ such that

$$\|x_n - z_n\| = \|[x_n]\|_{\tilde{X}} < C$$

for all n. Hence the sequence $(x_n - z_n)$ is bounded in X. Now, for all n,

$$\tilde{K}([x_n]) = K(x_n) = K(x_n) - 0 = K(x_n) - K(z_n) = K(x_n - z_n)$$

since $z_n \in N(K)$. But then, by compactness of K, there is a subsequence $K(x_{n_k})$ that converges (in Y). Hence $\tilde{K}([x_{n_k}])$ converges in Y, which is what we wished to show.

Now, observe that $\tilde{K}$ is compact, onto Y and is 1-to-1. Hence $\tilde{K}$ is bounded (hence closed), onto Y and is 1-to-1. But then there exists $C > 0$ such that for all $\tilde{x} \in \tilde{X}$,

$$(*) \;\; \|\tilde{x}\|_{\tilde{X}} \leq C \left\|\tilde{K}(\tilde{x})\right\|_Y$$

We're now ready to prove that Y is finite dimensional. We'll proceed by contradiction.

Suppose that $\dim Y = \infty$. But then there exists a sequence (y_n) in Y such that $\|y_n\|_Y = 1$ for all n, and for all $i \neq j$, one has $\|y_n - y_m\|_Y > 1/2$. See solution 4 above for construction of such a sequence. Clearly then (y_n) has no convergent subsequence in Y. Since $\tilde{K}$ is 1-to-1, let $x_n = \tilde{K}^{-1}(y_n)$. But then by (*) above, (x_n) is a bounded sequence in $\tilde{X}$, and, as noted above, its image, namely (y_n), does not have a convergent subsequence. This contradicts compactness of $\tilde{K}$. □

Exersize 17. Show that if X is an infinite dimensional normed vector space, then there is a sequence $\{x_n\}$ such that $\|x_n\| = 1, \|x_n - x_m\| \geq 1$ for $m \neq n$.

Solution 17. For convenience, S and Sp will be used as in solution 4. Throughout, all our vectors are taken to be nonzero.

Now, let $x_1 \in X$. But then there exists $y_1 \in X$ such that $\dim Sp(x_1, y_1) = 2$. But then, according to solution 3, there exists $x_2 \in Sp(x_1, y_1)$ such that $x_2 \in S$ and $\|x_1 - x_2\| \geq 1$. Proceed inductively. At the n^{th} step, since X is infinite dimensional, there exists $y_n \in X$ such that $\dim Sp(x_1, x_2, \ldots, x_n, y_n) = n + 1$. So there exists $x_{n+1} \in S$ such that $\|x_i - x_{n+1}\| \geq 1$ for $1 \leq i \leq n$. Thus (x_n) is the sequence we need. □

Exersize 18. Show that every finite-dimensional vector space can be made into a strictly convex normed vector space.

Solution 18. Observe that both, $\mathbb{R}^n$ and $\mathbb{C}^n$ are strictly convex vector spaces (under the standard Euclidean norm). This follows easily from the Cauchy-Schwartz inequality, since both are actually Hilbert spaces. Now, if $\dim V = n$, where V is a vector space over $K = \mathbb{R}, \mathbb{C}$, then V is isomorphic, as a vector space, to K^n. Suppose that $f : K^n \to V$ realizes this isomorphism (for instance, if $\{\beta_1, \ldots, \beta_n\}$

is a basis for V, we could take $f(e_j) = \beta_j$), where $\{e_1, \ldots, e_n\}$ is the standard basis for K^n, and extend f linearly to all of K^n). Now define a norm on V via $\|v\| = E(f^{-1}(v))$, where E is the standard Euclidean norm on K^n. Then V is strictly convex under $\|\cdot\|$. □

Exersize 19. Let V be a vector space having n linearly independent elements $v_1, \ldots, v_n$ such that every element $v \in V$ can be expressed in the form (4.12). Show that $\dim V = n$.

Solution 19. Obviously $\dim V \leq n$, since V is spanned by n vectors. Say $u_1, \ldots, u_m$ also span V. Then we can write

$$\begin{array}{rcl} v_1 &=& c_1^1 u_1 + c_1^2 u_2 + \cdots + c_1^m u_m \\ \vdots & \vdots & \vdots \\ v_n &=& c_n^1 u_1 + c_n^2 u_2 + \cdots + c_n^m u_m \end{array}$$

We also know that each u_j is a linear combination of $v_1, \ldots, v_n$. Hence we may write

$$\begin{array}{rcl} v_1 &=& c_1^1 \sum_{j=1}^n d_1^j v_j + c_1^2 \sum_{j=1}^n d_2^j v_j + \cdots + c_1^m \sum_{j=1}^n d_m^j v_j \\ \vdots & \vdots & \vdots \\ v_n &=& c_n^1 \sum_{j=1}^n d_1^j v_j + c_n^2 \sum_{j=1}^n d_2^j v_j + \cdots + c_n^m \sum_{j=1}^n d_m^j v_j \end{array}$$

Or, equivalently,

$$\begin{array}{rcl} 0 &=& (c_1^1 d_1^1 - 1)v_1 + \sum_{j=2}^n c_1^1 d_1^j v_j + \sum_{j=1}^n c_1^2 d_2^j v_j + \cdots + \sum_{j=1}^n c_1^m d_m^j v_j \\ \vdots & \vdots & \vdots \\ 0 &=& \sum_{j=1}^n c_n^1 d_1^j v_j + \sum_{j=1}^n c_n^2 d_2^j v_j + \cdots + \sum_{j=1}^{n-1} c_n^m d_m^j v_j + (c_n^m d_m^n - 1)v_n \end{array}$$

If $m < n$, then the system above clearly has at least one nontrivial solution in $c_i^j d_k^l$, contradicting linear independence of $v_1, \ldots, v_n$. Hence $m \geq n$. Thus $\dim V \geq n$. We conclude that $\dim V = n$. □

Exersize 20. Let V, W be subspaces of a Hilbert space. If $\dim V < \infty$ and $\dim V < \dim W$, show that there is a $u \in W$ such that $\|u\| = 1$ and $(u, v) = 0$ for all $v \in V$.

Solution 20. Suppose that $u_1, \ldots, u_n$ is a basis for V. We may assume, without loss of generality, that $\{u_1, \ldots, u_n\}$ is an orthonormal basis (perform Gramm-Schmidt). Now, obviously $V \neq W$, so there exists $w \in W$ such that $w \notin V$. Hence the set $\{u_1, \ldots, u_n, w\}$ is linearly independent. Perform Gramm-Schidt orthonormalization process on w to obtain the set $\{u_1, u_2, \ldots, u_n, u\}$ of orthonormal vectors. Then $u \notin V$, $\|u\| = 1$ and $(u, u_j) = 0$ for $1 \leq j \leq n$. Then, obviously by linearity of the inner product, $(v, u) = 0$ for all $v \in V$. We're done. □

Exersize 21. For X, Y Banach spaces, let A be an operator in $B(X, Y)$ such that $R(A)$ is closed and infinite-dimensional. Show that A is not compact.

Solution 21. Well, since Y is a Banach space and $R(A)$ is closed, $R(A)$ is also a Banach space. Let $\tilde{Y} = R(A)$. Then $A \in B(X, \tilde{Y})$ and is onto. Now use solution 7 above. □

Exersize 22. Suppose X is a Banach space consisting of finite linear combinations of a denumerable set of elements. Show that $\dim X < \infty$.

Solution 22. We prove here the contrapositive. So, suppose that X is infinite dimensional. Suppose $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ is a basis for X in the sense that for any $k \in \mathbb{N}$ and any choice of $i_1, i_2, \ldots, i_k \in \mathbb{N}$, $x_{i_1}, \ldots, x_{i_k}$ are linearly independent and every element of X is a finite linear combination of elements of S. We may assume, without loss of generality, that $\|x_j\| = 1$, for $j \in \mathbb{N}$ (simply normalize x_j: $x_j = x_j / \|x_j\|$, by abuse of notation). Now let

$$s_n = \sum_{j=1}^{n} \frac{1}{2^j} x_j$$

Then the sequence (s_n) is easily seen to be Cauchy (it is bounded above by partial sums of a geometric series), but does not converge, since

$$\sum_{j=1}^{\infty} \frac{1}{2^j} x_j$$

cannot be written as a linear combination of finitely many elements of S, by linear independence of the elements of S. Hence X is not Banach. This proves the contrapositive of the original claim. □

Exersize 23. Show that every linear functional on a finite dimensional normed vector space is bounded.

Solution 23. Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $\|\cdot\|_K$ on K^n be the norm given by

$$\|(\alpha_1, \ldots, \alpha_n)\| = \sum_{j=1}^{n} |\alpha_j|$$

Suppose now that f is a linear functional on K^n. Then we have

$$f((\alpha_1, \ldots, \alpha_n)) = \alpha_1 f((1, 0, \ldots, 0)) + \cdots + \alpha_n f((0, \ldots, 0, 1))$$

hence

$$|f((\alpha_1, \ldots, \alpha_n))| = \left| \sum_{j=1}^{n} \alpha_j f((\delta_{1j}, \ldots, \delta_{nj})) \right| \leq \max_{1 \leq j \leq n} \{\|f((\delta_{1j}, \ldots, \delta_{nj}))\|\} \sum_{j=1}^{n} |\alpha_j|$$

$$= \max_{1 \leq j \leq n} \{\|f((\delta_{1j}, \ldots, \delta_{nj}))\|\} \|(\alpha_1, \ldots, \alpha_n)\|$$

so f is bounded.

Now, suppose that X is an n-dimensional vector space over K. Then X is isomorphic, as a vector space, to K^n. Let $\phi : V \to K^n$ realize this isomorphism. Define a norm on V by $\|v\|_V = \|\phi(v)\|$. Then it is easily seen that every functional g on V is bounded with respect to this norm, with $\|g\|_V = \left\|g \circ \phi^{-1}\right\|$, where $g \circ \phi^{-1}$ is viewed as a functional on K^n. But all norms on V are equivalent, hence g is bounded with respect to any norm on V. □

Exersize 24. If M is a subspace of a Banach space X, then we define

$$\text{codim} M = \dim X/M$$

If M is closed, show that $\text{codim} M = \dim M^\circ$ and $\text{codim} M^\circ = \dim M$.

Solution 24. For convenience, set $\tilde{X} = X/M$. Suppose now that $\dim \tilde{X} = n < \infty$. Let $[\beta_1], \ldots, [\beta_n]$ be a basis for $\tilde{X}$. Then, as is easily verified, $\beta_1, \ldots, \beta_n$ are linearly independent in X. In fact, we may write

$$X = Sp(\beta_1, \ldots, \beta_n) \oplus M$$

Let $\tilde{X}^*$ and X^* denote the spaces of functionals on $\tilde{X}$ and X, respectively. Consider

$$\phi : \tilde{X}^* \to X^*$$

given by

$$\phi(f)(\beta_j) = f([\beta_j])$$

and extending linearly to all of X by setting $\phi(f)(x) = 0$ for all $x \notin Sp(\beta_1, \ldots, \beta_n)$. Then clearly ϕ is a vector space homomorphism, and $\phi(\tilde{X}^*) \subset M^\circ$. Since $\dim \tilde{X}^* = \dim X = n$, and $N(\phi) = 0$, we conclude that $\dim \phi(\tilde{X}^*) = n$, so $\dim M^\circ \geq n$.

On the other hand, if $g \in M^\circ$, then g is completely determined by its action on $\beta_1, \ldots, \beta_n$. Let $\tilde{g}$ be the functional on $\tilde{X}$ given by

$$\tilde{g}([\beta_j]) = g(\beta_j)$$

Then, obviously, $\phi(\tilde{g}) = g$. So in fact ϕ is a vector space isomorphism from $\tilde{X}^*$ to M°. Hence $\dim M^\circ = \dim \tilde{X}^* = n$.

If $\dim \tilde{X} = \infty$, then for all $n \in N$ there exists a subspace $V_n \subset \tilde{X}$ of dimension n. Restricting ϕ to V_n shows that $\dim M^\circ \geq n$. This shows that $\dim M^\circ = \infty$.

One shows similarly that $\text{codim} M^\circ = \dim M$. □

Exersize 25. If M, N are subspaces of a Banach space X and $\text{codim} N < \dim M$, show that $M \cap N \neq \{0\}$.

Solution 25. We prove the contrapositive. Suppose that $M \cap N = \{0\}$. Assume first that $\dim M = m < \infty$. Let $\beta_1, \ldots, \beta_m$ a basis for M. By assumption, $\beta_1, \ldots, \beta_m \notin N$, and $[\beta_1], \ldots, [\beta_m]$ are linearly independent when viewed as elements of X/N. Hence

$$Sp([\beta_1], \ldots, [\beta_m]) \subset X/N$$

is a subspace of X/N of dimension m. So $\text{codim} N \geq \dim M$.

Suppose that $\dim M = \infty$. But then for any $n \in N$ there exists a subspace V_n of dimension n in M, for which the above argument can be carried out. Hence again $\text{codim} N \geq n$ for all $n \in \mathbb{N}$, so $\text{codim} N = \dim M = \infty$.

This proves the contrapositive. □

Exersize 26. If $A \in B(X, Y), R(A)$ is dense in Y and D is dense in X, show that A maps D onto a dense subset of Y.

Solution 26. Why do we need the assumption $R(A)$ is dense in X? We know that A is bounded, hence continuous from X to Y, when X and Y are viewed as topological spaces (the topologies are induced by the norms). But a continuous map maps dense sets to dense sets. To see this, suppose $f : U \to V$ is continuous, with U, V topological spaces. Say $D \subset U$ is dense. Let $v \in V$ and N an open neighborhood about v in V. By continuity of f, $f^{-1}(N)$ is open in U. By density of D, there exists $d \in D$ such that $d \in f^{-1}(N)$. Hence $f(d) \in f(D)$ and $f(d) \in N$. This shows that $f(D)$ is dense in V. We're done. □

Exersize 27. Show that every infinite dimensional normed vector space has an unbounded linear functional.

Solution 27. Let $S = \{x_1, x_2, \dots x_n, \dots\} \subset X$ be linearly independent. We can assume, without loss of generality, that $\|x_j\| = 1$ after normalizing each x_j. Let V be the subspace of X consisting of all finite linear combinations of elements of S. Define f_j on V by

$$f_j(x) = \begin{cases} jc & \text{if} \quad x = cx_j \\ 0 & \text{otherwise} \end{cases}$$

Let f be the functional on V defined by

$$f = \sum_{j=1}^{\infty} f_j$$

f is well-defined, since for each $x \in V$, f is a finite sum. Now, f is unbounded on V (it is unbounded on the unit ball in V). To see this, let $x = x_1 + x_2 + \cdots + x_n$. Extend f to all of X by defining $f(x) = 0$ when $x \notin V$. $\square$

Exersize 28. Let F, G be linear functionals on a vector space V, and assume that $F(v) = 0 \implies G(v) = 0$, $v \in V$. Show that there is a scalar C such that $G(v) = CF(v)$, $v \in V$.

Solution 28. From the hypothesis it is evident that $N(F) \subset N(G)$. We claim that $\operatorname{codim} N(F) \leq 1$. So, suppose that $\operatorname{codim} N(F) > 0$ and suppose that V is a vectorspace over the field K. Then $F : V \to K$ is a homomorphism of vector spaces which is surjective. But then

$$\tilde{F} : V/N(F) \to K$$

is an isomorphism of vector spaces, where $\tilde{F}$ is induced by F in a natural way (i.e., $\tilde{F}([v]) = F(v)$). Hence $\dim V/N(F) = \dim K = 1$.

From this it follows that F is completely determined by its action on $\beta \notin N(F)$. If $N(F) = V$, then there is nothing to prove. Otherwise, since $N(F) \subset N(G)$ and, by the argument above, $\operatorname{codim} N(G) \leq 1$. If $N(G) = V$, then again we're done (take $C = 0$). If not, then there exists $\beta \notin$N(F), N(G). Hence both F and G are completely determined by their action on β. Since K is a field and $F(\beta) \neq 0$, we may take $C = G(\beta)/F(\beta)$. $\square$

Exersize 29. If $\{x_k\}$ is a sequence of elements in a normed vector space X and $\{\alpha_k\}$ is a sequence of scalars, show that a necessary and sufficient condition for the existence of an $x' \in X'$ satisfying

$$x'(x_k) = \alpha_k \text{ and } \|x'\| = M,$$

is that

$$\left| \sum_1^n \beta_k \alpha_k \right| \leq M \left\| \sum_1^n \beta_k x_k \right\|$$

holds for each n and scalars $\beta_1, \dots, \beta_n$.

Solution 29. Suppose there exists such a functional x'. Let $\beta_1, \dots, \beta_n$ be scalars, fixed. Let $x = \beta_1 x_1 + \cdots + \beta_n x_n \in X$. Applying x' to x, we obtain

$$x'(x) = x' \left(\sum_{j=1}^n \beta_j x_j \right) = \sum_{j=1}^n \beta_j \alpha_j$$

Since $\|x'\| = M$, we immediately get from above

$$\left|\sum_{j=1}^{n} \beta_k \alpha_k\right| = |x'(x)| \leq M \|x\| = M \left\|\sum_{j=1}^{n} \beta_k x_k\right\|$$

Let now V be the vector subspace of X consisting of all finite linear combinations of $\{x_k\}$. Suppose that

$$\left|\sum_{1}^{n} \beta_k \alpha_k\right| \leq M \left\|\sum_{1}^{n} \beta_k x_k\right\|$$

holds for each n and scalars $\beta_1, \ldots, \beta_n$.

Define $x'(x_k) = \alpha_k$ and extend linearly to V. Then extend x' to all of X by defining $x'(y) = 0$ if $y \notin V$.

However, the condition that

$$\left|\sum_{1}^{n} \beta_k \alpha_k\right| \leq M \left\|\sum_{1}^{n} \beta_k x_k\right\|$$

holds for each n and scalars $\beta_1, \ldots, \beta_n$ does not imply that $\|x'\| = M$. Indeed, if $K > M$, then one has

$$\left|\sum_{1}^{n} \beta_k \alpha_k\right| \leq K \left\|\sum_{1}^{n} \beta_k x_k\right\|$$

holds for each n and scalars $\beta_1, \ldots, \beta_n$. □

Exersize 30. If a normed vector space X has a subspace M such that M and X/M are complete, show that X is a Banach space.

Solution 30. □

Exersize 31. If X, Y are normed vector spaces and $B(X, Y)$ is complete, show that Y is complete.

Solution 31. Let $\{y_n\}$ be a Cauchy sequence in Y. Fix nonzero x_0 in X. Define $A_n(cx_0) = cy_n$ and for all $x \notin Sp(x_0)$, let $A(x) = 0$. Clearly then $A_n \in B(X, Y)$ and $\|A_n\| = \|y_n\| / \|x_0\|$ (the norms are taken in respective spaces, of course). Now, observe that

$$\|(A_n - A_m)(x_0)\| = \|A_n(x_0) - A_m(x_0)\| \leq \|y_n - y_m\| / \|x_0\|$$

So $\{A_n\}$ is Cauchy in $B(X, Y)$, hence $A_n \to A \in B(X, Y)$. But then $A_n(x_0) \to A(x_0)$ in Y. Hence $y_n \to A(x_0)$ in Y. So Y is complete, hence Banach. □

Exersize 32. Prove that if $\{T_n\}$ is a sequence in $B(X, Y)$ such that $\lim T_n x$ exists for each $x \in X$, then there is a $T \in B(X, Y)$ such that $T_n x \to Tx$ for all $x \in X$.

Solution 32. Set $Tx = \lim T_n x$. First observe that T is linear. Indeed, $T(cx+y) = \lim T_n(cx+y) = c \lim T_n(x) + \lim T_n(y) = cTx + Ty$. Next observe that T is bounded.

Suppose first that $T_n(x) \to T(x)$ uniformly for all $x \in S$, where S is the unit sphere in X. If this is the case, then there exists $N \in \mathbb{N}$ such that for all x of unit norm,

$$\|Tx - T_n x\| < 1 \text{ for all } n \geq N, x \in S$$

so

$$\|Tx\| < T_N x + 1, x \in S$$

hence

$$\|Tx\| \leq \|T_N\| + 1 \text{ for all } x \in S$$

So T is bounded.

If convergence on S is not necessarily uniform, then T is not necessarily bounded. Consider the following example.

Let $S = \{e_1, e_2, \dots\}$, where $e_j = (x_k^j) : \mathbb{N} \to \{0, 1\}$ given by $x_k^j = \delta_{kj}$. We can visualize each e_j as an ∞-tuple, having 1 in j^{th} position and 0 everywhere else.

Let V be the (infinite-dimensional) vector space over $\mathbb{R}$ formed by all *finite* linear combinations of elements of S. Endow V with the norm

$$\|c_1 e_{j_1} + c_2 e_{j_2} + \cdots + c_m e_{j_m}\| = \sum_1^m |c_k|$$

Define $\tilde{T}_j : V \to \mathbb{R}$ by $\tilde{T}_j(e_i) = j\delta_{ji}$, and extend $\tilde{T}_j$ linearly to V. Then each $\tilde{T}_j$ is a linear functional on V. Define

$$T_n = \sum_1^n \tilde{T}_k$$

Then each T_n is obviously a linear functional on V. Let

$$T = \lim_{n\to\infty} T_n = \sum_{n=1}^{\infty} \tilde{T}_n$$

T is well-defined, since for each $x \in V$, $T(v)$ is a finite sum (recall that v is a *finite* linear combination of elements of S). Further, each T_n is bounded. T, however, is unbounded! Indeed,

$$|T(e_1 + e_2 + \cdots + e_n)| = \left|\sum_{j=1}^n \tilde{T}_j(e_j)\right| = \sum_{j=1}^n j = \frac{n(n+1)}{2}$$

On the other hand,

$$\|e_1 + e_2 + \cdots + e_n\| = n$$

So

$$\frac{|T(e_1 + e_2 + \cdots + e_n)|}{\|e_1 + e_2 + \cdots + e_n\|} = \frac{n+1}{2}$$

which cannot be bounded by a constant.

Exersize 33. Let F, G be linear functionals on a vector space V, and assume that $F(v) = 0 \implies G(v) = 0$, $v \in V$. Show that there is a scalar C such that $G(v) = CF(v)$, $v \in V$.

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holds for each n and scalars $\beta_1, \ldots, \beta_n$.

Solution 34. Suppose there exists such a functional x'. Let $\beta_1, \ldots, \beta_n$ be scalars, fixed. Let $x = \beta_1 x_1 + \cdots + \beta_n x_n \in X$. Applying x' to x, we obtain

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$$|T(e_1 + e_2 + \cdots + e_n)| = \left|\sum_{j=1}^n \tilde{T}_j(e_j)\right| = \sum_{j=1}^n j = \frac{n(n+1)}{2}$$

On the other hand,

$$\|e_1 + e_2 + \cdots + e_n\| = n$$

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