5.10. Zero-sum Games and Linear Programming

We start with a zero-sum game G with matrix A. The value of G can obtained from

$$\begin{aligned} v(G) &= \max\left\{\min\left\{\vec{p}^T A\vec{q} : \vec{q} \in \Delta_{n-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\ &= \min\left\{\max\left\{\vec{p}^T A\vec{q} : \vec{p} \in \Delta_{m-1}\right\} : \vec{q} \in \Delta_{n-1}\right\} \end{aligned}$$

So the second player tries to keep the value of the game low, while the second player tries to make this value as large as possible. Now the first player cannot make the value of the game higher than w if the second player presents him with a vector $A\vec{q}$ for which all the entries are less than or equal to w. Indeed, in this case

$$\begin{aligned} \vec{p}^T(A\vec{q}) &\le \vec{p}^T \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} \\ &= (p_1, \cdots, p_n)\begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} \\ &= wp_1 + \cdots + wp_n \\ &= (p_1 + \cdots + p_n)w \\ &= 1 \cdot w = w \end{aligned}$$

In other words, the second player tries to solve the following linear programming problem:

Minimize w

Subject to the conditions

$$A\vec{q} \le \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}$$

$$\vec{q} \in \Delta_{n-1}$$

5.10.1. Proposition.

Let G be a zero-sum game. Then the value of the game can be obtained as solution of the following linear programming problem:

Minimize w

Subject to the conditions

$$A\vec{q} \le \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}$$

$$\vec{q} \in \Delta_{n-1}$$

Proof. Let z be the solution of the linear programming problem. Hence there is a value $\vec{q}_o \in \Delta_{n-1}$ so that $A\vec{q}_o \le (z,\cdots,z)^T$. As we have seen above, this implies that for every value of $p \in \Delta_{m-1}$ we have $\vec{p}^T A\vec{q}_o \le z$, i.e. $\max\left\{p^T A\vec{q}_o : \vec{p} \in \Delta_{m-1}\right\} \le z$. We conclude that

$$\begin{aligned} v &= \min\left\{\max\left\{\vec{p}^T A\vec{q} : \vec{p} \in \Delta_{m-1}\right\} : \vec{q} \in \Delta_{n-1}\right\} \\ &\le \max\left\{\vec{p}^T A\vec{q}_0 : \vec{p} \in \Delta_{m-1}\right\} \\ &\le z \end{aligned}$$

Hence $v \le z$. Conversely, let $\varepsilon > 0$ be given. Then

$$v = \min\left\{\max\left\{\vec{p}^T A\vec{q} : \vec{p} \in \Delta_{m-1}\right\} : \vec{q} \in \Delta_{n-1}\right\} < v + \varepsilon$$

Hence there is a vector $\vec{p}_1 \in \Delta_{m-1}$ so that

$$\max\left\{\vec{p}^T A\vec{q}_1 : \vec{p} \in \Delta_{m-1}\right\} < v + \varepsilon$$

If the vector $A\vec{q}_1$ had any entry larger than $v + \varepsilon$, then we could find a unit vector $\vec{e}_i$ so that $\vec{e}_i^{\,T} A\vec{q}_1 \ge v + \varepsilon$. This would imply that $v + \varepsilon \le \max\left\{\vec{p}^T A\vec{q}_1 : \vec{p} \in \Delta_{m-1}\right\} < v + \varepsilon$, a contradiction. Hence $\vec{q}_1$ satisfies

$$A\vec{q}_1 \le \begin{pmatrix} v+\varepsilon \\ \vdots \\ v+\varepsilon \end{pmatrix}$$

$$\vec{q}_1 \in \Delta_{n-1}$$

Therefore the solution z of the linear programming problem cannot be larger than $v + \varepsilon$, i.e.

$$z \le v + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $z \le v$. Both inequalities together yield $z = v$, and the proof is complete.

Since the roles of both players are "dual" to each other, and since $\vec{p}^T A = \left(A^T \vec{p}\right)^T$, we easily can modify the proof of the last theorem to obtain

5.10.2. Proposition.

Let G be a zero-sum game. Then the value of the game can be obtained as solution of the following linear programming problem:

Maximize w

Subject to the conditions

$$A^T \vec{p} \ge \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}$$

$$\vec{p} \in \Delta_{m-1}$$

The last two propositions show that the value of a game can be obtained from solutions of a linear programming problem. However, it is not clear at this point what the corresponding point of equilibrium would be.

5.10.3. Theorem.

Let G be a zero-sum game with value v with matrix A. If $\vec{q}_o$ is any solution of

$$\begin{aligned}&\text{Minimize } w\\&\text{Subject to the conditions}\\&A\vec{q} \le \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}\\&\vec{q} \in \Delta_{n-1}\end{aligned}$$

and if $\vec{p}_o$ is any solution of

$$\begin{aligned}&\text{Maximize } w\\&\text{Subject to the conditions}\\&A^T\vec{p} \ge \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}\\&\vec{p} \in \Delta_{m-1}\end{aligned}$$

then $(\vec{p}_o, \vec{q}_o)$ is a saddle point and hence a point of equilibrium.

Proof. We know already that the value v of the game can be obtained from either one of the two linear programming problems. Hence

$$A\vec{q}_o \le \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}$$

$$A^T\vec{p}_o \ge \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}$$

Hence for any $\vec{p} = (p_1, \cdots, p_m) \in \Delta_{m-1}$ and any $\vec{q} = (q_1, \cdots, q_n) \in \Delta_{n-1}$ we have

$$\vec{p}^T A\vec{q}_o \le v(p_1 + \cdots + p_m) = v$$

$$\vec{p}_0 A\vec{q} \ge v(q_1 + \cdots + q_n) = v$$

and this is exactly the property required in the definition of saddle points.

The last theorem suggests studying linear programming problem and their solution. This will lead to a discussion of the simplex algorithm. Before we start translating the problem of finding a saddle point into linear programming problems in standard form, we given an example:

Example. Let G be the zero-sum game given by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We would like to find all points of equilibrium and the value of the game. There are two linear programming problems to solve:

Minimize w

$$\text{Subject to } \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \\ t \end{pmatrix} \le \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

$$q + r + s + t = 1$$

$$q, r, s, t \ge 0$$

This leads to the inequalities

$$q + r \le w$$
$$2r \le w$$
$$s \le w$$
$$s + t \le w$$

Since we also know that

$$2w \ge q + r + s + t = 1$$

it follows that the smallest possible value is $w = \frac{1}{2}$. Therefore the value of the game is 0.5. The values for q,r,s,t can be computed from

$$q + r = \frac{1}{2}$$

$$r \le \frac{1}{4}$$

$$s + t = \frac{1}{2}$$

Hence $(q,r,s,t) = \left(\frac{1}{2} - r, r, \frac{1}{2} - t, t\right)$ where $0 \le r \le \frac{1}{4}$ and $0 \le t \le \frac{1}{2}$.

In order to find all points of equilibrium, we also have to solve:

$$\text{Maximize } w$$

$$\text{Subject to} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} l \\ m \\ n \\ p \end{pmatrix} \geq \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

$$l+m+n+p=1$$

$$l,m,n,p \geq 0$$

We already know that $w=\frac{1}{2}$, and hence have to solve

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \\ p \end{pmatrix} \geq \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$l+m+n+p=1$$

$$l,m,n,p \geq 0$$

Therefore

$$l \geq \frac{1}{2}$$

$$l+2m \geq \frac{1}{2}$$

$$n+p \geq \frac{1}{2}$$

$$p \geq \frac{1}{2}$$

$$l+m+n+p=1$$

$$l,m,n,p \geq 0$$

The second and the third inequality are redundant, since they are implied by the first and forth inequality. Hence

$$l \geq \frac{1}{2}$$

$$p \geq \frac{1}{2}$$

$$l+m+n+p=1$$

$$l,m,n,p \geq 0$$

This leads to the only possible solution $(l,m,n,p)=\left(\frac{1}{2},0,0,\frac{1}{2}\right)$. Hence saddle points are all combinations of $(l,m,n,p)=\left(\frac{1}{2},0,0,\frac{1}{2}\right)$ with a point of the form $(q,r,s,t)=\left(\frac{1}{2}-r,r,\frac{1}{2}-t,t\right)$ where $0\le r\le\frac{1}{4}$ and $0\le t\le\frac{1}{2}$.

In general, there are various standard forms of linear programming problems. Two are particularly useful in our context, namely

$$\begin{aligned}&\text{Maximize } \vec{a}\cdot\vec{x}\\&\text{subject to}\\&M\vec{x}\le\vec{b}\\&\vec{x}\ge 0\end{aligned}$$

or

$$\begin{aligned}&\text{Minimize } \vec{a}\cdot\vec{x}\\&\text{subject to}\\&M\vec{x}\ge\vec{b}\\&\vec{x}\ge 0\end{aligned}$$

where M is a matrix. It is not difficult to write our two linear programming problems that resulted from the problem of finding points of equilibrium in this form. As an example, we show how to rewrite the problem

$$\begin{aligned}&\text{Minimize } w\\&\text{Subject to the conditions}\\&A\vec{q}\le\begin{pmatrix}w\\ \vdots\\ w\end{pmatrix}\\&\vec{q}\in\Delta_{n-1}\end{aligned}$$

in one of the standard forms just mentioned. First, since we do not know whether w is positive, we write w in t he form

$$w=v-u$$

with $u,v\ge 0$. Instead of finding a minimum for w, we now try to find a maximum for the positive number $-w=u-v$:

Maximize $u-v$

Subject to the conditions

$$A\vec{q} \le \begin{pmatrix} v-u \\ \vdots \\ v-u \end{pmatrix}$$

$$\vec{q} \in \Delta_{n-1}$$

Hence the problem is equivalent to

Maximize $(0,\cdots,0,1,\text{-}1)(q_1,\cdots,q_n,u,v)$

Subject to the conditions

$$\begin{pmatrix} a_{11} & \dots & a_{1n} & 1 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & 1 & -1 \\ 1 & \cdots & 1 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ u \\ v \end{pmatrix} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$q_1,\cdots,q_n,u,v \ge 0$$

Similarly, the problem

Maximize w

Subject to the conditions

$$A^T\vec{p} \ge \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}$$

$$\vec{p} \in \Delta_{m-1}$$

is equivalent to

Minimize $(0,\cdots,0,1,\text{-}1)(p_1,\cdots,p_n,u,v)$

Subject to the conditions

$$\begin{pmatrix} a_{11} & \dots & a_{m1} & 1 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & 1 & -1 \\ 1 & \cdots & 1 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ u \\ v \end{pmatrix} \ge \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$q_1,\cdots,q_n,u,v \ge 0$$

Note that the matrices occurring in both linear programming problems are transposes of each other.