5.9. Symmetric Games

Symmetric games are the games that are being considered as fair. The roles of both players are completely symmetric. They have the same strategies available. If player 1 wins by using strategy A against player 2, who is using B, then player 1 will loose by playing strategy B against player 2's strategy A. Mathematically, this can be expressed as follows:

5.9.1. Definition.

A two player zero-sum game G with matrix A is called symmetric, if A is skew symmetric, i.e. if $A = -A^T$

An example of a symmetric game is, of course, "Rock, Paper, Scissor".

5.9.2. Proposition.

Every symmetric game G has value 0.

Proof. We compute:

$$
\begin{aligned}
v(G) &= \max\left\{\min\left\{\vec{p}^T A\vec{q} : \vec{q} \in \Delta_{m-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\
&= \max\left\{\min\left\{\left(\vec{p}^T A\vec{q}\right)^T : \vec{q} \in \Delta_{m-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\
&= \max\left\{\min\left\{\left(\vec{q}^T A^T \vec{p}\right)^T : \vec{q} \in \Delta_{m-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\
&= \max\left\{\min\left\{\left(-\vec{q}^T A\vec{p}\right)^T : \vec{q} \in \Delta_{m-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\
&= -\min\left\{\max\left\{\left(-\vec{q}^T A\vec{p}\right)^T : \vec{q} \in \Delta_{m-1}\right\} : \vec{p} \in \Delta_{m-1}\right\} \\
&= -v(G)
\end{aligned}
$$

Hence $v(G) = 0$.

5.9.3. Proposition.

Let G be a symmetric game. Then $\left(\vec{p}_e, \vec{q}_e\right)$ *is a point of equilibrium if and only if*

$$
\begin{aligned}
A\vec{p}_e &\le \vec{0} \\
A\vec{q}_e &\le \vec{0}
\end{aligned}
$$

Proof. Assume first that the two conditions hold. Then $A\vec{p}_e \le \vec{0}$ implies that

$$\begin{aligned}\vec{p}_e{}^T A &= \left(A^T \vec{p}_e\right)^T \\ &= \left(-A\vec{p}_e\right)^T \\ &= -\left(A\vec{p}_e\right)^T \\ &\ge 0^T\end{aligned}$$

and hence, since all coordinates of $\vec{q}_e$ are positive, we conclude that $\vec{p}_e A\vec{q}_e \ge 0$. Also, since all coordinates of $\vec{p}_e$ are positive and since $A\vec{q}_e \le \vec{0}$, it follows that $\vec{p}_e A\vec{q}_e \le 0$, i.e.

$$\vec{p}_e A\vec{q}_e = 0$$

If $\vec{q} \in \Delta_{m-1}$ is given, then all coordinates of $\vec{q}$ are positive. Since all coordinates of $\vec{p}_e{}^T A$ are also positive, we find that

$$\vec{p}_e A\vec{q} \ge 0 = \vec{p}_e A\vec{q}_e$$

Similarly, if $\vec{p} \in \Delta_{m-1}$ is given, then

$$\vec{p} A\vec{q}_e \le 0 = \vec{p}_e A\vec{q}_e$$

and therefore $\left(\vec{p}_e, \vec{q}_e\right)$ is a point of equilibrium.

Conversely, if $\left(\vec{p}_e, \vec{q}_e\right)$ is a point of equilibrium, then

$$\vec{p}_e{}^T A\vec{q}_e = \max\left\{\vec{p}^T A\vec{q}_e : \vec{p} \in \Delta_{m-1}\right\} = 0$$

If at least one coordinate of $A\vec{q}_e$ were strictly positive then there would be a (unit) vector $\vec{p} \in \Delta_{m-1}$ so that $\vec{p}^T A\vec{q}_e > 0$, contradicting the fact that $\max\left\{\vec{p}^T A\vec{q}_e : \vec{p} \in \Delta_{m-1}\right\} = 0$. Hence

$$A\vec{q}_e \le \vec{0}$$

Similarly, we show that

$$A\vec{p}_e \le \vec{0}$$

Using the last theorem, it is now easy to find all saddle points of “rock, paper, and scissor”. We have to find all solutions of

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r + s + t = 1$$

$$r, s, t \ge 0$$

This set of equations leads to

$$
\begin{aligned}
t &\le s \\
r &\le t \\
s &\le r \\
r + s + t &= 1 \\
r, s, t &\ge 0
\end{aligned}
$$

The only solution to this set of equations is given by $r = s = t = \frac{1}{3}$, and therefore the only point of equilibrium obtained from $\vec{p}_e = \vec{q}_e = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.