5.7. Randomized 2 x 2 Bimatrix Games

We can now completely solve randomized bimarix games, if both matrices have two rows and two columns. So let us start with two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

We can actually explicitly compute the maps β_1 and β_2. The elements of Δ_1 are pairs $(r, 1-r)$, where $0 \le r \le 1$. If we want to find the value of $\beta_2(r, 1-r)$, then we have to proceed as follows: First, we compute the value of $(x_1, x_2) = (r, 1-r)B$:

$$(x_1, x_2) = (r, 1-r)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$x_1 = rb_{11} + (1-r)b_{21}$$

$$x_2 = rb_{12} + (1-r)b_{22}$$

Then we find the maximal value of x_1 and x_2:

1. If $x_1 < x_2$, then $\beta_2(r, 1-r) = \left\{(0,1)^T\right\}$
2. If $x_1 > x_2$, then $\beta_2(r, 1-r) = \left\{(1,0)^T\right\}$
3. If $x_1 = x_2$, then $\beta_2(r, 1-r) = \left\{(s, 1-s)^T : 0 \le s \le 1\right\}$.

Now there are three options:

1. $b_{11} \ge b_{12}$ and $b_{21} \ge b_{22}$ (in this case, we say that the first column of *B* dominates the second column of *B*). Then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \beta_2(r, 1-r)$$

   for all choices of *r*.
2. $b_{11} \le b_{12}$ and $b_{21} \le b_{22}$ (in this case, we say that the second column of *B* dominates the first column of *B*). Then

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \beta_2(r, 1-r)$$

   for all choices of *r*.
3. Neither of the previous two cases applies. In this case, we consider the two lines given by the equations $f_1(r) = rb_{11} + (1-r)b_{21}$ and $f_2(r) = rb_{12} + (1-r)b_{22}$ At the endpoint *r = 0*, we find $f_1(0) = b_{21}$ and $f_2(0) = b_{22}$, while at the other

endpoint we find $f_1(1) = b_{11}$ and $f_2(1) = b_{12}$. If the columns do not dominate each other, then either $f_1(0) < f_1(0)$ and $f_1(1) > f_2(1)$, or else $f_1(0) > f_1(0)$ and $f_1(1) < f_2(1)$. In either case, there will be a number *r* between 0 and 1 so the $f_1(r) = f_2(r)$. For this value of *r*, we find that

$$\beta_2(r, 1-r) = \left\{(s, 1-s)^T : 0 \le s \le 1\right\}$$

Analogous statements hold for the map β_2, only that we now have to consider the rows of *A* instead of the columns of *B*. This leads to the following table for the map $\beta = \beta_1 \times \beta_2$.

	Row 1 dominates Row 2 of A	Row 2 dominates row 1 of A	No row domination for A
Column 1 dominates Column 2 of B	For all values of *r,s* $\beta((r,1-r),(s,1-s)) = \{(1,0),(1,0)\}$	For all values of *r,s* $\beta((r,1-r),(s,1-s)) = \{(0,1),(1,0)\}$	$(\forall r)((1,0) \in \beta_2(r,1-r))$
Column 2 dominates Column 2 of B	For all values of *r,s* $\beta((r,1-r),(s,1-s)) = \{(1,0),(0,1)\}$	For all values of *r,s* $\beta((r,1-r),(s,1-s)) = \{(0,1),(0,1)\}$	$(\forall r)((0,1) \in \beta_2(r,1-r))$
No column domination for B	$(\forall s)((1,0) \in \beta_1(s,s-1))$	$(\forall s)((0,1) \in \beta_1(s,s-1))$	$\exists(r,s)(\beta(r,s) = \Delta_2 \times \Delta_2)$

From this table, we can read off the points of equilibrium:

	Row 1 dominates Row 2 of A	Row 2 dominates row 1 of A	No row domination for A
Column 1 dominates Column 2 of B	Equilibrium at (1,0) and (0,1)	Equilibrium at (0,1) and (0,1)	Equilibrium at $\vec{p}$, (1,0) where $\vec{p} \in \beta_1(1,0)$
Column 2 dominates Column 2 of B	Equilibrium at (1,0) and (0,1)	Equilibrium at (0,1) and (0,1)	Equilibrium at $\vec{p}$, (0,1) where $\vec{p} \in \beta_1(0,1)$
No column domination for B	Equilibrium at (1,0), $\vec{q}$ where $\vec{q} \in \beta_2(1,0)$	Equilibrium at (0,1), $\vec{q}$ where $\vec{q} \in \beta_2(0,1)$	$\exists(r,s)(\beta(r,s) = \Delta_2 \times \Delta_2)$

Example. Find the points of equilibrium, if

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

- Neither matrix has dominating rows or columns. So we have to find values of *r* and *s* so that

$$(r,1-r)B=(x,x)$$

$$A\begin{pmatrix} s \\ 1-s \end{pmatrix}=\begin{pmatrix} y \\ y \end{pmatrix}$$

This means that

$$r-2(1-r)=-r+3(1-r)$$

$$s-(1-s)=2s+4(1-s)$$

or

$$r=\frac{5}{7}$$

$$s=\frac{4}{5}$$

Hence the point of equilibrium is $\left(\frac{5}{7},\frac{2}{7}\right)$ and $\left(\frac{4}{5},\frac{1}{5}\right)$.

Example. Find the points of equilibrium, if

$$A=\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

$$B=\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

The second row of *A* dominates the first row. So the optimal strategy for the first player is *(0,1)*, no matter which strategy the second player is using. This means that the second player has to find a best response to the best strategy of the first player. He therefore has to maximize

$$(0,1)B\begin{pmatrix} s \\ 1-s \end{pmatrix}=(-2,3)\begin{pmatrix} s \\ 1-s \end{pmatrix}$$

The maximum occurs for $s = 0$. Hence the point of equilibrium is at *(0,1)* (for player 1) and *(0,1)* (for player 2).

The idea of row or column domination can be used to treat bimatrix games of higher dimensions. Simply remove repreatedly rows of *A* that are dominated by other rows of *A* and columns of *B* that are dominated by other columns of *B*. The resulting bimatrix game might be much easier to treat.