5.6. Nash's Theorem

John Nash proved the following theorem:

5.6.1. Theorem

Let G be a randomized bimatrix game. Then G has an equilibrium.

In order to prove this theorem, we are asked to find a fixed point of the map β. In order to do this, we apply Kakutani's Fixed Point Theorem. We will state a version of this theorem below. The proof is too complicated to be repeated here.

5.6.2. Theorem (Kakutani, 1941)

Let C be a compact convex subset of Euclidean n – space. Let F be a map that assigns to each $x \in C$ *a closed convex subset* $F(x) \subseteq C$. *If for each open set* $U \subseteq \Re^n$ *the set* $\{x \in C : F(x) \subseteq U\}$ *is open in C, then there is a point* x_0 *so that* $x_0 \in F(x_0)$

We will apply this theorem to the map β. So we have to verify that for each open set U the set $\left\{(\vec{p},\vec{q}) \in \Delta_m \times \Delta_n : \beta(\vec{p},\vec{q}) \in U\right\}$ is open in $\Delta_m \times \Delta_n$. Actually, we will prove a stronger statement:

5.6.3. Lemma

If $(\overrightarrow{p_0},\overrightarrow{q_0}) \in \Delta_m \times \Delta_n$ *is given, then the sets*

$$U_1 = \left\{\vec{p} \in \Delta_m : \beta_1(\vec{p}) \subseteq \beta_1(\overrightarrow{p_0})\right\}$$

and

$$U_2 = \left\{\vec{q} \in \Delta_n : \beta_2(\vec{q}) \subseteq \beta_2(\overrightarrow{q_0})\right\}$$

are open.

Proof. Let A and B be the matrices defining the bimatrix game. We apply the results from the interlude: If

$$(x_1,\cdots,x_n) = \overrightarrow{p_0}^T B$$
$$\mu_0 = \max\{x_1,\cdots,x_n\}$$
$$J_0 = \{j : x_j = \mu_0\}$$

then

$$\beta_2\left(\overrightarrow{p_0}\right)=\left\{\vec{q}\in\Delta_{n-1}:\sum_{j\in J_0}q_j=1\right\}$$

Pick a number $\varepsilon>0$ so that $x_i<\mu_0-\varepsilon$ whenever $i\notin J_0$. Since matrix multiplication is continuous, there is a number $\delta>0$ so that $\|\overrightarrow{p_0}-\vec{p}\| < \delta$ implies $\|\overrightarrow{p_0}^T B-\vec{p}^T B\|<\varepsilon/2$. If $(y_1,\cdots,y_n)=\vec{p}^T B$, then $i\notin J_0$ implies that

$$\begin{aligned}y_i&=x_i+y_i-x_i\\&\le x_i+\varepsilon/2\\&<(\mu_0-\varepsilon)+\varepsilon/2\\&=\mu_0-\varepsilon/2\end{aligned}$$

while $i\in J_0$ implies

$$\begin{aligned}y_i&=x_i+y_i-x_i\\&=\mu+y_i-x_i\\&\ge\mu-\varepsilon/2\end{aligned}$$

It follows that the maximum of the entries of $(y_1,\cdots,y_n)=\vec{p}^T B$ has to be greater than or equal to $\mu_0-\varepsilon/2$ and can occur only on indices belonging to J_0, If

$$\mu=\max\{y_1,\cdots,y_n\}$$
$$J=\{j:x_j=\mu\}$$

then $J\subseteq J_0$ and therefore

$$\begin{aligned}\beta_2\left(\vec{p}\right)&=\left\{\vec{q}\in\Delta_{n-1}:\sum_{j\in J}q_j=1\right\}\\&\subseteq\left\{\vec{q}\in\Delta_{n-1}:\sum_{j\in J_0}q_j=1\right\}\\&=\beta_2\left(\overrightarrow{p_0}\right)\end{aligned}$$