5.5. Case Study: Fixed Points for the Game of Rock, Paper, Scissors

The matrices for the game of rock, paper and scissors were given by

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Since this is a zero-sum game, it does not matter whether we look for equilibria in the sense of definition A or in the sense of definition B. Using the notation of the last section, for each choice of $(p_1, p_2, p_3) \in \Delta_2$, we first have to compute

$$\begin{aligned}(x_1, x_2, x_3) &= (p_1, p_2, p_3)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \\ &= (p_2 - p_3, p_3 - p_1, p_1 - p_2) \\ \mu &= \max\{p_2 - p_3, p_3 - p_1, p_1 - p_2\} \\ J &= \{j : x_j = \mu\} \\ \beta_2(\vec{p}) &= \left\{(q_1, q_2, q_3) \in \Delta_2 : \sum_{j \in J} q_j = 1\right\}\end{aligned}$$

This leads to various cases:

1. $p_1 > \frac{1}{3}$ and $p_2 < \frac{1}{3}$. In this case,

$$\begin{aligned}3p_1 &> p_1 + p_2 + p_3 \\ 2p_1 &> p_2 + p_3 \\ p_1 - p_2 &> p_3 - p_1\end{aligned}$$

and

$$\begin{aligned}3p_2 &< p_1 + p_2 + p_3 \\ 2p_2 &< p_1 + p_3 \\ p_2 - p_3 &< p_1 - p_2\end{aligned}$$

Hence

$$\mu = p_1 - p_2$$
$$J = \{3\}$$
$$\beta_2(p_1,p_2,p_3) = \left\{(q_1,q_2,q_3)\in\Delta_2 : \sum_{j\in J} q_j = 1\right\}$$
$$= \{(q_1,q_2,q_3)\in\Delta_2 : q_3 = 1\}$$
$$= \{(0,0,1)\}$$

2. $p_2 > \frac{1}{3}$ and $p_3 < \frac{1}{3}$. Proceeding as in the previous case, we find that
$$\beta_2(p_1,p_2,p_3) = \{(1,0,0)\}$$
3. $p_3 > \frac{1}{3}$ and $p_1 < \frac{1}{3}$. In this case, we find that
$$\beta_2(p_1,p_2,p_3) = \{(0,1,0)\}$$

Those three cases actually cover all possibilities in which all of the number $p_i \neq \frac{1}{3}$. If $p_1 > \frac{1}{3}$ and $p_2 > \frac{1}{3}$, then $p_1 + p_2 + p_3 = 1$ implies that $p_3 < \frac{1}{3}$, and we are in case 2. Similarly, $p_1 < \frac{1}{3}$ and $p_3 < \frac{1}{3}$ leads to $p_2 > \frac{1}{3}$, and we are again in case 2.

So we are left with the possibility that either one or all of the numbers p_i are equal to 1/3. Let us first consider the case where exactly one of the numbers p_i equals 1/3: Assume that $p_1 = \frac{1}{3}$. If $p_2 > \frac{1}{3}$, then $p_3 < \frac{1}{3}$, and we are back in case 2. Hence we may assume that $p_2 < \frac{1}{3}$ and $p_3 > \frac{1}{3}$. In this case, we can compute that $p_1 - p_2 = p_3 - p_2 > p_2 - p_3$ and therefore $J = \{2,3\}$. Hence

4. $p_1 = \frac{1}{3}$ and $p_2 < \frac{1}{3}, p_3 > \frac{1}{3}$ lead to
$$\beta_2(p_1,p_2,p_3) = \{(0,r,1-r) : 0 \le r \le 1\}$$
5. $p_2 = \frac{1}{3}$ and $p_1 > \frac{1}{3}, p_3 < \frac{1}{3}$ lead to
$$\beta_2(p_1,p_2,p_3) = \{(r,0,1-r) : 0 \le r \le 1\}$$
6. $p_3 = \frac{1}{3}$ and $p_1 < \frac{1}{3}, p_2 > \frac{1}{3}$ lead to
$$\beta_2(p_1,p_2,p_3) = \{(r,1-r,0) : 0 \le r \le 1\}$$

Finally, last case deals with the possibility that all numbers p_i equal 1/3:

7. $p_1 = p_2 = p_3 = \frac{1}{3}$. In this case, $J = \{1,2,3\}$ and

$$\beta_2(p_1, p_2, p_3) = \Delta_2$$

Since the problem is symmetric for both players, we find that $\beta_1 = \beta_2$. We define

$$\beta = \beta_1 = \beta_2$$

What does this function say about points of equilibrium? The vectors (p_1, p_2, p_3) and (q_1, q_2, q_3) are at equilibrium precisely if

$$(p_1, p_2, p_3) \in \beta(q_1, q_2, q_3)$$
$$(q_1, q_2, q_3) \in \beta(p_1, p_2, p_3)$$

Clearly, since $\beta\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \Delta_2$, the vectors *(1/3,1/3,1/3)* and *(1/3,1/3,1/3)* are at equilibrium. Can there be others?

If at least one coordinate of (q_1, q_2, q_3) is different from 1/3, then all the vectors in $\beta(q_1, q_2, q_3)$ have at least one coordinate equal to 0.

Moreover

$$\beta(0, q_2, q_3) \subseteq \{(p_1, p_2, 0) : 0 \le p_1, p_2, p_1 + p_2 = 1\}$$
$$\beta(q_1, 0, q_3) \subseteq \{(0_1, p_2, p_3) : 0 \le p_2, p_3, p_2 + p_3 = 1\}$$
$$\beta(q_1, q_2, 0) \subseteq \{(p_1, 0, p_3) : 0 \le p_1, p_3, p_1 + p_3 = 1\}$$

This implies that

$$(p_1, p_2, p_3) \in \beta(q_1, q_2, q_3)$$

has one coordinate equal to zero, and then

$$(q_1, q_2, q_3) \in \beta(p_1, p_2, p_3)$$

has a different coordinate equal to 0. Using $(p_1, p_2, p_3) \in \beta(q_1, q_2, q_3)$ again, (p_1, p_2, p_3) has a second coordinate equal to 0 and therefore is a unit vector.
With the same argument, (q_1, q_2, q_3) is also a unit vector, and we already know that there is no equilibrium consisting of pure strategies.