5.4. Interlude: Maxima on Simplices

If we would like to use fixed points to find points of equilibrium for randomized matrix games, we have to know more about the structure of the sets of the form $\beta(x,y)$. We trace back the definitions to find a description of $\beta(x,y)$

The function $\omega_1(\vec{p},\vec{q})$ is given by

$$\omega_2(\vec{p},\vec{q}) = \vec{p}^{\,T} B\overline{q}$$

The Lagrange function $\lambda_2(\vec{p})$ is defined as

$$\lambda_2(\vec{p}) = \max\left\{\vec{p}^{\,T} B\vec{q} : \vec{q} \in \Delta_{n-1}\right\}$$

Now the expression $\vec{p}^{\,T} B$ represents a fixed row-vector $\vec{x} = (x_1,\cdots,x_n)$. We are asked to compute

$$\begin{aligned} M &= \max\left\{x_1q_1 + \cdots + x_nq_n : 0 \le q_1,\cdots,q_n \text{ and } q_1 + \cdots + q_n = 1\right\} \\ &= \max\left\{\vec{x}\cdot\vec{q} : \vec{q} \in \Delta_{n-1}\right\} \end{aligned}$$

This task is actually not too hard: Let

$$\mu = \max\left\{x_1,\cdots,x_n\right\}$$

Then for each choice of $(q_1,\cdots,q_n) \in \Delta_{n-1}$ we have

$$x_1q_1 + \cdots + x_nq_n \le \mu q_1 + \cdots + \mu q_n = \mu(q_1 + \cdots + q_n) = \mu$$

and hence

$$M \le \mu$$

Conversely, let *k* be an index so that $x_k = \mu$ and let e_k be the *k*th unit vector in *n* dimensions. Then $e_k \in \Delta_{n-1}$ and $\mu = \vec{x}\cdot e_k \le M$. Hence

$$M = \mu$$

But we can do better than that. Let $J(\vec{x}) = \left\{k : x_k = \max\left\{x_1,\cdots,x_n\right\}\right\}$. Then

$$\begin{aligned} &\vec{x}\cdot\vec{q} = M \text{ if and only if } q_i = 0 \text{ whenever } i \notin J(\vec{x}) \\ &\text{if and only if } \sum_{j\in J(\vec{x})} q_j = 1 \end{aligned}$$

Indeed, if $q_i = 0$ whenever $i \notin J(\vec{x})$, then

$$\begin{aligned} x_1 q_1 + \cdots + x_n q_n &= \sum_{j \in J(\vec{x})} x_j q_j \\ &= \sum_{j \in J(\vec{x})} \mu q_j \\ &= \mu \sum_{j \in J(\vec{x})} q_j \\ &= \mu \cdot 1 = M \end{aligned}$$

Conversely, if $\vec{x} \cdot \vec{q} = M$ and if $q_i > 0$ for at least one $i \notin J(\vec{x})$, then we obtain the contradiction

$$\begin{aligned} M &= x_1 q_1 + \cdots + x_i q_i + \cdots + x_n q_n \\ &< x_1 q_1 + \cdots + \mu q_i + \cdots + x_n q_n \text{ (since } q_i > 0 \text{ and } x_i < \mu) \\ &\le \mu q_1 + \cdots + \mu q_i + \cdots + \mu q_n \\ &= \mu \end{aligned}$$

Putting things back together leads to the following: If

$$\begin{aligned} (x_1, \cdots, x_n) &= \vec{p}^T B \\ \mu &= \max\{x_1, \cdots, x_n\} \\ J &= \{j : x_j = \mu\} \end{aligned}$$

then

$$\beta_2(\vec{p}) = \left\{ \vec{q} \in \Delta_{n-1} : \sum_{j \in J} q_j = 1 \right\}$$

These sets are known as faces[8] of Δ_{n-1}. Draw some pictures! All of them are closed, convex sets spanned by some of the vertices of Δ_{n-1}. It follows that for each pair $(\vec{p}, \vec{q}) \in \Delta_{m-1} \times \Delta_{n-1}$ the set $\beta(\vec{p}, \vec{q}) = \beta_1(\vec{q}) \times \beta_2(\vec{p}) \subseteq \Delta_{m-1} \times \Delta_{n-1}$ is a face of the convex set $\Delta_{m-1} \times \Delta_{n-1}$

In the next section, we will attempt to use these ideas for the game of "rock, paper and scissors".

[8] In general, a subset $F \subseteq C$ of a convex set C is called a face of C if it is convex itself, and if in addition for each choice of $0 < \lambda < 1$ and $x, y \in C$ the relation $\lambda x + (1 - \lambda) y \in F$ implies that $x, y \in F$