5.3. Mixed Strategies and 2-Player Games in General

5.3.1. Definition.

A two-player game G is a quadruple $(X, Y, \omega_1, \omega_2)$ *where*

$$\omega_1, \omega_2 : X \times Y \to \mathbb{R}$$

are two real-valued functions.
The set X is called the strategy set of player 1, the set Y is called the strategy set of player 2, and the function ω_j *is called the outcome function*[7] *of the game for player j.*

Typical examples for two-player games are bimatrix games. In this case, *X* is the set of all row numbers and *Y* is the set of all column numbers of the matrices defining the bimatrix game. If (A, B) are the two $m \times n$-matrices defining the bimatrix game, and if

$$A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$$
$$B = (b_{ij})_{1 \le i \le m, 1 \le j \le n}$$

then

$$\omega_1(i, j) = a_{ij}$$
$$\omega_2(i, j) = b_{ij}$$

Bimatrix games have an important extension to mixed strategies. In order to define those, we need the standard simplices:

5.3.2. Definition.

Let n be a fixed positive integer. The (n-1)-dimensional standard simplex is defined as

$$\Delta_{n-1} = \left\{ (p_1, \cdots, p_n) \in \mathbb{R}^n : 0 \le p_1, \cdots, 0 \le p_n, p_1 + \cdots + p_n = 1 \right\}$$

The elements of Δ_{n-1} can be viewed as probability distributions. Note that the elements of Δ_{n-1} have *n* coordinates, so they have to viewed as probability distributions on sets with *n* elements. The choice of the index *n-1* in Δ_{n-1} is a matter of taste and a source for endless discussions.

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[7] In most books the functions ω_1 and ω_2 are called payoff-functions. I don't wish to be paid off.

5.3.3. Definition.

Let G be a bimatrix game with $m \times n$ -matrices $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ *and* $B = (b_{ij})_{1 \le i \le m, 1 \le j \le n}$. *The randomization of G is the game*

$$G^* = (\Delta_n, \Delta_m, \omega_1^*, \omega_2^*)$$

where

$$\omega_1^*((p_1, \cdots, p_n), (q_1 \cdots q_m)) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} p_i q_j$$

$$\omega_2^*((p_1, \cdots, p_n), (q_1 \cdots q_m)) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} p_i q_j$$

If we write the elements of Δ_m and Δ_n as column vectors, then

$$\omega_1^*(\vec{p}, \vec{q}) = \vec{p}^T A \vec{q}$$

$$\omega_2^*(\vec{p}, \vec{q}) = \vec{p}^T B \vec{q}$$

We now define equilibria of games as follows:

5.3.4. Definition

Let $G = (X, Y, \omega_1, \omega_2)$ *be a two-player game. A point of equilibrium of G is a pair of elements* $(x_e, y_e) \in X \times Y$ *such that the following two conditions are satisfied:*

$$(\forall x \in X) \;\; \omega_1(x, y_e) \le \omega_1(x_e, y_e)$$

$$(\forall y \in y) \;\; \omega_2(x_e, y) \le \omega_2(x_e, y_e)$$

We can reformulate this definition in terms of fixed points. We start with the following notations:

5.3.5. Definition.

Let $G = (X, Y, \omega_1, \omega_2)$ *be a two-player game. We define the Lagrange functions*

$$\lambda_1 : X \to \mathbb{R} \cup \{\infty\}$$

$$\lambda_2 : Y \to \mathbb{R} \cup \{\infty\}$$

by

$$\lambda_1(y) = \max\{\omega_1(x, y) : x \in X\}$$

$$\lambda_2(x) = \max\{\omega_2(x, y) : y \in Y\}$$

For each $x \in X$ *and each* $y \in Y$ *let*

$$\beta_1(y) = \{x \in X : \omega_1(x,y) = \lambda_1(y)\}$$
$$\beta_2(x) = \{y \in Y : \omega_2(x,y) = \lambda_2(x)\}$$

and

$$\beta(x,y) = \beta_1(y) \times \beta_2(x)$$

The following result follows immediately from the definitions:

5.3.6. Proposition.

Let $G = (X, Y, \omega_1, \omega_2)$ be a two-player game. An element $(x_e, y_e) \in X \times Y$ is a point of equilibrium if and only if it is a fixed point of β in the sense that

$$(x_e, y_e) \in \beta(x_e, y_e)$$