5.2. Bimatrix Games

Bimatrix games are games that are played by two players. Each player has a different set of preference, and we assume that each player numbers his or her preferences in a list. Outcomes that are less desirable have low numbers. So each player prefers outcomes with high numbers. What is desirable for one player might not be desirable for the other player. For example, if colors are to be ordered, the first player might order them as red < blue < orange, and the second player could order them as blue < orange < red. So the number 1 means "red" to the first player and "blue" to the second player. This coding of preferences by number is frequently referred to as a "utility function". So if the outcomes of a game are colors, and if

$$\begin{pmatrix} red & blue & orange & red \\ blue & orange & red & blue \\ red & red & blue & orange \end{pmatrix}$$

presents the strategic form of a game, then we would have to code this as

$$\begin{pmatrix} (1,3) & (2,1) & (3,2) & (1,3) \\ (2,1) & (3,2) & (1,3) & (2,1) \\ (1,3) & (1,3) & (2,1) & (3,2) \end{pmatrix}$$

Two matrices really best represent this situation, one for each player:

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 3 & 3 & 1 & 2 \end{pmatrix}$$

This leads to the following definition:

5.2.1. Definition.

An $n \times m$ bimatrix game is a pair of matrices (A_1, A_2), each having n rows and m columns.
A row index is also called a pure strategy for the first player, and a column index is called a pure strategy for the second player.
The entry in row k and column l of the matrix A_i is denoted by $\omega_i(l,k)$ and called the outcome for player i under the strategies (k,l). Hence

$$A_1 = (\omega_1(k,l))_{1 \le k \le n, 1 \le l \le m}$$

$$A_2 = (\omega_2(k,l))_{1 \le k \le n, 1 \le l \le m}$$

Both players are of course trying to optimize their outcomes. There are certain pure strategies that lead both players to believe that they better not change their strategy. Those are the Nash equilibria. We give two different definitions of equilibria. Even though those definitions seem to express different ideas, they are equivalent mathematically.

5.2.2. Definition A

Let (A_1, A_2) *be a bimatrix game. A pair (k,l) of pure strategies is called an equilibrium if the following two properties are satisfied:*

$$\omega_1(k, j) \geq \omega_1(k,l) \text{ for all indices } j$$

$$\omega_2(i,l) \geq \omega_2(k,l) \text{ for all indices } i$$

This definition might be expressed as follows: Suppose that the first player wants to use strategy *k*. She now carries out a worst case analysis and computes how much she can get out of the game under the paranoid assumption that her opponent tries to minimize her results. In this sense, the equilibria do not represent optimal outcomes of games. Rather, they serve as estimates for damage control.

The game of rock, scissors and paper gives an example of a game without equilibria. The prisoner's dilemma is a game with an equilibrium that does not give a solution that is optimal for both players.

A bimatrix game can have more than one equilibrium. For example, if

$$A_1 = A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

then the pairs *(1,1)* and *(2,2)* are equilibria. Both do not lead to an optimal outcome of the game for either player. The strategies *(1,2)* and *(2,1)* would lead to better outcome for both players. Why don't they choose two players one of the later strategies? What do they have to lose? Is their decision derived from the fear that the other player could do something bad to them? Equilibria, at least with this definition, serve as security blankets. They result from paranoia. The following alternative way to define equilibria makes this even clearer:

Player 1, instead of trying to maximize his outcome, now strives to minimize the outcome of his opponent, and likewise, the second player tries to minimize the outcome of first player. Mathematically, swapping the outcome functions and adding negative signs express this:

$$\omega_1' = -\omega_2$$

$$\omega_2' = -\omega_1$$

Hence the definition of equilibria now has the following form:

$$\omega_1'(i,l) \le \omega_1'(k,l) \text{ for all indices } i$$
$$\omega_2'(k,j) \le \omega_2'(k,l) \text{ for all indices } j$$

5.2.3. Definition B [6] (Nash, Cournot)

Let (A_1, A_2) be a bimatrix game. A pair (k,l) of pure strategies is called an equilibrium if the following two properties are satisfied:

$$\omega_1(i,l) \le \omega_1(k,l) \text{ for all indices } i$$
$$\omega_2(k,j) \le \omega_2(k,l) \text{ for all indices } j$$

This is actually the definition preferred by many authors. Both definitions are identical for zero-sum games, since $\omega_1 = -\omega_2$ implies that $\omega_i' = \omega_i$.

The second definition expresses the following procedure: Suppose that first player already knows that the second player wants to use strategy *l*. He then chooses, among all his strategies, the one the promises the highest outcome. The second player acts accordingly.

If we use the procedure just outlined by the bimatrix game given by

$$A_1 = A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

then we end up with the bimatrix game defined by

$$A_1' = A_2' = \begin{pmatrix} 0 & -2 \\ -2 & -1 \end{pmatrix}$$

Note that the best strategies have not changed, and now at least one of them leads to an optimal outcome for both players. The paranoia vanished in a puff of logic?

We should also remark that the outcomes of games might be different for different equilibria. This could not happen for the zero-sum games discussed previously.

We have to be careful if we translate one definition into the other definition: The bimatrix game with matrices $A_1 = A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ have two equilibria in the sense of definition A, and two different equilibria in the send of definition B.

Which definition should we prefer? The first definition was more useful for games in which Zermelo's algorithm was used. The second form of the definition will be more useful for the mathematical treatment of matrix games.

6 Nash suggested this definition around 1950. However, an earlier version can be already found by A.A. Cournot in "Recherché sur les principes mathématiques de la théorie de la richesse, Paris 1838"

Those two different definitions seem to be plays with words inside the game of game theory. Beware of the used car salesman of economics!

5.2.4. Summary

1. Equilibria can be defined for bimatrix games in general. They are considered to be "solutions" to a game.
2. Equilibria do not need to lead to optimal solutions.
3. Equilibria do not always exists, and, if they exist, they might not be unique.
4. Different equilibria can lead to different outcomes of the game.
5. There are two different ways to define equilibria. The definitions can be translated into each other by an easy algebraic manipulation.
6. For zero-sum games, both ways to define equilibria agree. Also, in this case, there might be more than one equilibrium (or non at all), but they lead to the same outcome (see next section).