5. Games on Matrices

5.1. Examples of Matrix Games

In all examples of the games discussed up to this point, all players had complete information about the current game situation. This will not always be the case. Chance move can obscure the game and make it harder to move according to preset strategies. Sometimes, the game becomes so involved that we really do not know anymore in which situation we currently find ourselves. This can happen in many different ways. In this section, we will study games where all players move simultaneously without knowing the other players moves. In this case, game trees can no longer be drawn, and we have to rely completely on strategic forms of games.

5.1.1. Example (Rock, Paper, Scissors)

The game of "Rock, Paper, and Scissors" is a typical example of a two-player game where both players have to move simultaneous. On a count of three, both players show symbols for a rock (closed fist), paper (open hand) or a pair of scissors (two fingers). The rules are: rock beats scissors, scissors beat paper, and paper beats rock. This can be represented by the following table, which also gives the strategic form of the game:

	Rock	Paper	Scissors
Rock	(Draw, Draw)	(Lose, Win)	(Win, Lose)
Paper	(Win, Lose)	(Draw, Draw)	(Lose, Win)
Scissors	(Lose, Win)	(Win, Lose)	(Draw, Draw)

Now player 1 chooses a row and player 2 chooses a column. Both players make their choices simultaneously. The entry in the table indicates who won and who lost.

We now can again code the outcomes: a 1 represents "Win", "Lose" is indicated by a -1, and "Draw" is indicated by a 0. We obtain the following game matrix:

$$\begin{pmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix}$$

Since the sums of all the entries are always equal to 0, we could call this again a zero-sum game, and list only the first entries:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Each player has three "pure strategies", namely picking one of the rows (player I) or columns (player 2). Always playing this strategy leads to no result: A player either always wins, always looses, or the game always ends up in a tie. So the players should use "mixed strategies", i.e. randomize their choices of "pure strategies". They could roll a dye. If the dye shows 1 or 3, they would use row (or column) 1, if the dye shows 2 or 4, they would use the second row (or column), and a 3 or 6 would lead the choice of the last row (column).

The minimum of each row is -1, and the maximum value of those numbers is also -1. So, using only pure strategies, the first player would expect to loose the game, and so would the second player. Hence both players would expect to always lose? Unlikely.

What would the minimax theorem say in this case? As we just observed, the maximum of the minimum values of each row is -1. If we compute the minimum value of all maximum values of the columns, we obtain the value +1. So in this case, the minimax and the maximin are different numbers.

Does the matrix have saddle points[3]? An entry r_{ij} of a matrix was called a saddle point, if all other entries in row i are at least as large, and all other entries in column j are no larger. So the matrix of "Rock, Paper, and Scissors" does not have a saddle point.

What happens if we used mixed strategies, and what is the best way to pick a mixed strategy? The first player would randomize her strategies by picking row 1 with probability r_1, row 2 with probability r_2 and row 3 with probability r_3. Of course, this would mean that $r_1, r_2, r_3 \geq 0$ and $r_1 + r_2 + r_3 = 1$. Accordingly, the second player would play one of his columns with probabilities $c_1, c_2, c_3 \geq 0$ with $c_1 + c_2 + c_3 = 1$.

What would this mean for winning and loosing. If the game is played *N* times, then player 1 would use "Rock" $r_1 N_1$ many. How many times would she win by choosing "Rock". In this case, player 2 would have to chose "Scissors", and that would occur in c_3% of the times player 1 chooses "Rock", i.e. in $r_1 c_3 N$ many times the game is played. We find that the number of times player 1 wins is given by

$$\left(r_1 c_3 + r_2 c_1 + r_3 c_2\right) N$$

and she will loose in

$$\left(r_1 c_2 + r_2 c_3 + r_3 c_1\right) N$$

many games. The difference – her net result – is given by

$$\left(r_1 c_3 + r_2 c_1 + r_3 c_2 - r_1 c_2 - r_2 c_3 - r_3 c_1\right) N$$

many games. The coefficient of *N* can be expressed by matrix multiplication:

[3] We soon will have to be more careful with the terminology. Right now, saddle points and points of equilibria seem to be interchangeable, and there is no problem as long as we talk about zero-sum games. However, for games in general, there might be a difference between those two notions.

$$r = (r_1, r_2, r_3)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Multiplied by the number of times the game is played, this number will give the number of times player 1 win more than she looses, provided that both players player according to their "mixed strategies".

We can use a similar computation to find the net result of the second player. It is given by

$$\begin{aligned} c &= (r_1, r_2, r_3)\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= -r \end{aligned}$$

Note: The number resulting from the matrix multiplication is **not** the probability of winning the game. For example, the result of the matrix multiplication might be $r < 0$. Clearly, there are no negative probabilities. This number cannot even be used to compute the probability of winning, because we do not know the number of draws. If the game is played N times, then

1. Player 1 wins n times;
2. Player 1 looses m times;
3. There are k draws.

We find that

$$n - m = rN$$
$$n + m + k = N$$

The probability p of winning the game is

$$p = \frac{n}{N}$$

Since we have two equations and three unknowns, we cannot compute the values of n, m and k in terms of r and N. We have to discuss carefully what the numbers in mixed strategies want to tell us!

Abstractly, we arrived at the following setup:[4]

1. The pool of available mixed strategies for both players can be coded by the "standard simplices"
$$\Delta_2 = \left\{(x, y, z)^T : 0 \le x, y, z \text{ and } x + y + z = 1\right\}$$
2. If player 1 uses mixed strategy σ_1 and player 2 uses mixed strategy σ_2, then the net result for player 1 is given by

[4] Vectors are always written as column vectors. This explains the use of transposition in the following formulas.

$$\omega_1(\sigma_1,\sigma_2) = \sigma_1^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \sigma_2$$

while the net result for the second player is given by

$$\omega_2(\sigma_1,\sigma_2) = \sigma_1^T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sigma_2$$

Can we find saddle points now? How about points of equilibrium? What do we mean by those terms? We would like to express the fact the neither player has a reason to change the strategy. We could are looking for a pair of mixed strategies (σ_1,σ_2) so that for each other pair of mixed strategies (τ_1,τ_2) we have

$$\omega_1(\sigma_1,\sigma_2) \le \omega_1(\sigma_1,\tau_2)$$
$$\omega_2(\sigma_1,\sigma_2) \le \omega_2(\tau_1,\sigma_2)$$

We can write this as

$$\omega_1(\sigma_1,\sigma_2) = \min\{\omega_1(\sigma_1,\tau) : \tau \in \Delta_2\}$$
$$\omega_2(\sigma_1,\sigma_2) = \min\{\omega_2(\tau,\sigma_2) : \tau \in \Delta_2\}$$

Using the fact that the game of “Rocks, Paper, and Scissors” is a zero-sum game, we can reformulate the last equations as

$$\omega_1(\sigma_1,\sigma_2) = \min\{\omega_1(\sigma_1,\tau) : \tau \in \Delta_2\}$$
$$\omega_1(\sigma_1,\sigma_2) = \max\{\omega_1(\tau,\sigma_2) : \tau \in \Delta_2\}$$

Hence

$$\omega_1(\sigma_1,\sigma_2) = \min\left\{ \sigma_1^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tau : \tau \in \Delta_2 \right\}$$

$$\omega_1(\sigma_1,\sigma_2) = \max\left\{ \tau^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \sigma_2 : \tau \in \Delta_2 \right\}$$

Since

$$\tau^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \sigma_2 = \left[\tau^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \sigma_2 \right]^T$$

$$= \sigma_2^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}^T \tau$$

$$= -\sigma_2^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tau$$

we find that

$$\omega_1(\sigma_1, \sigma_2) = \max \left\{ \tau^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \sigma_2 : \tau \in \Delta_2 \right\}$$

$$= -\min \left\{ \sigma_2^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tau : \tau \in \Delta_2 \right\}$$

Therefore, we are looking for a pair of strategies $(\sigma_1, \sigma_2) \in \Delta_2 \times \Delta_2$ so that

$$\max \left\{ \sigma_1^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tau : \tau \in \Delta_2 \right\} = -\min \left\{ \sigma_2^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tau : \tau \in \Delta_2 \right\}$$

Arguments using three dimensional geometry show unique solution of this equation is given by

$$\sigma_1 = \sigma_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

So we indeed have a point of equilibrium for mixed strategies, and this point of equilibrium asked both players to use rock, scissors and paper at random, with equal probability. In this case, if the game is played a large number of times, both players will win 1/3 of the games and there will be a draw in the remaining 1/3.

5.1.2. Example (Prisoner's Dilemma)

Two prisoners are in separate cells. There is enough evidence to convict them of minor offence, giving them 1 year in prison each. However, if one of them finks, then he will be used as crown witness and go free, while the other prisoner will receive 3 years in prison. If both of the fink, then they cannot used as crown witnesses and in the end both receive 2 years in prison. What should they do?

We again start with a table:

	Quiet	Fink
Quiet	(1 year, 1 year)	(3 years, 0 years)
Fink	(0 years, 3 years)	(2 years, 2 years)

However, the years cannot be their preferences. Looking at the table, each comes up with his own preference functions:

First prisoner:

Best choice: 0 years in prison
Second choice: 1 year in prison
Third choice: 2 years in prison
Forth choice: 3 years in prison.

This leads to the following preference function:

$$u_1(fink, quiet) > u_1(quiet, quiet) > u_1(fink, fink) > u_1(quiet, fink)$$

The second prisoner has a different idea about preferences:

$$u_2(quiet, fink) > u_2(quiet, quiet) > u_2(fink, fink) > u_2(fink, quiet)$$

We might code those preferences with numbers[5]. Since higher numbers usually mean something good, we will be using the number of years not spend in prison as coding. The game matrix then will have following form, where the first row (column) corresponds to being quiet, and the second row (column) stands for finking:

$$\begin{pmatrix} (2,2) & (0,3) \\ (3,0) & (1,1) \end{pmatrix}$$

There is one equilibrium, obtained from both finking. However, both would be better off if they would keep their mouth shut.

In this example, both prisoners would have two pure strategies. What happens if the get to be in the same situation over and over again and randomize their actions? How many years would they now spend in prison in the average?

Each prisoner has his own matrix. They are $A_1 = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ for prisoner 1 and $A_2 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ for prisoner 2. The mixed strategies are now elements of

$$\Delta_1 = \{(r,s) : 0 \le r, s \text{ and } r + s = 1\}$$

The expected number of years not spent in prison for each prisoner would be given by

$$\omega_1(\tau_1, \tau_2) = \tau_1^T A_1 \tau_2$$

---

[5] Such a coding is often called a utility function.

and

$$\omega_2(\tau_1,\tau_2) = \tau_1^T A_2 \tau_2$$

So a point of equilibrium would be given by a pair of mixed strategies (σ_1,σ_2) so that for each other pair of mixed strategies (τ_1,τ_2) we have

$$\omega_1(\sigma_1,\sigma_2) \le \omega_1(\sigma_1,\tau_2) = \sigma_1^T A_1 \tau_2$$

$$\omega_2(\sigma_1,\sigma_2) \le \omega_2(\tau_1,\sigma_2) = \tau_1^T A_2 \sigma_2$$

We can write this as

$$\omega_1(\sigma_1,\sigma_2) = \min\{\omega_1(\sigma_1,\tau) : \tau \in \Delta_1\}$$

$$\omega_2(\sigma_1,\sigma_2) = \min\{\omega_2(\tau,\sigma_2) : \tau \in \Delta_1\}$$

We now use the fact that the second matrix is the transpose of the first matrix. Hence we are looking for strategies (σ_1,σ_2) so that

$$\min\{\sigma_1^T A_1 \tau : \tau \in \Delta_1\} = \min\{\sigma_2^T A_1 \tau : \tau \in \Delta_1\}$$

In this case, we can assume that $\sigma_1 = \sigma_2$. Hence, for each given value of $\sigma_1 = (r,1-r)$ we have to compute the value of $\min\{\sigma_1^T A\tau : \tau \in \Delta_1\}$. So for each fixed value of r between 0 and 1, we are asked to minimize

$$\begin{aligned}(r,1-r)\begin{pmatrix}2 & 0\\ 3 & 1\end{pmatrix}\begin{pmatrix}s\\ 1-s\end{pmatrix} &= 2rs + 1(1-r)(1-s) + 3(1-r)s\\ &= 2rs + 1 - r - s + rs + 3s - 3rs\\ &= 1 - r + 3s\end{aligned}$$

Clearly the minimum occurs at $s = 0$ and has the value $1 - r$. The maximal number not spent in prison would occur for $r = 0$, and the probability that the prisoners are quiet should be equal to 0. Hence, in this case, mixing up the strategies does not help. The optimum strategies for both prisoners are to fink.

The last example shows that there are limits to the theory we are developing. They might find solutions that seem to be acceptable to the players, but are far from optimal.