4.3. Strategies for Games with Chance Moves

We would like to define games for *n* players that allow also chance moves. So we introduce an extra player (the treacherous lady luck), who is responsible for the chance moves. This new player gets number 0. All the other players behave as before.

4.3.1. Definition.

An n-player game G with chance moves and outcomes E_1 ,…, E_n consists of

1. *A tree t with nodes E and leaves L.*
2. *A function* $p: E \setminus L \to \{0,1,\ldots,n\}$ *that labels each node that is not also a leaf of the tree t with one of the numbers 0,1,…,n.*
3. *A function* $\omega: L \to E_1 \times \cdots \times E_n$ *that labels each leaf of the tree t with one possible combination of outcomes.*
4. *For each edge starting at a node N with p(N) = 0 a probability distribution* δ_N *on the set of all edges starting at node N.*

*If node n is labeled with i, then we say that it is the i*th *player's turn to move at node n.*

4.3.2. Definition.

Let G be an n – player game with chance moves.

1. *A strategy* σ_i *for the i*th *player is a function that assigns to each node N where it is the i*th *player's turn to move a probability function* δ_N *on the set of all edges starting at node N.*
2. *The strategy* σ_i *is called a pure strategy, if the probability functions* δ_N *determined by the i*th *player take only the values 0 and 1.*

In other words, a pure strategy tells the player at each node what to do. Strategies that are not pure lead to some sort of mixed strategies.

If each player chooses a strategy, then we can combine all the strategies to obtain a compound lottery:

4.3.3. Definition.

Let G be an n – player game with chance moves. For each i let σ_i *be a strategy for the i*th *player.*

1. *The strategies* σ_i*, together with the probability distributions for the chance moves, determine a compound lottery* $\lambda(\sigma_1,\cdots,\sigma_n)$ *on the game tree of G.*

2. *If all the outcomes E_i for the players consist of real numbers, then i*th *component ω_i of ω is a random variable on the leaves of the game tree. This random variable has an expected value $E(\omega_i)$. The outcome of the game with respect to the strategies $(\sigma_1, \cdots, \sigma_n)$ is defined as*

$$\omega(\sigma_1, \cdots, \sigma_n) = (E(\omega_1), \cdots, E(\omega_n))$$

Values of games can be again computed by using Zermelo's algorithm. Also, Zermelo's algorithm can be used to determine optimal strategies for the players. These optimal strategies necessarily will be pure strategies, unless we break ties in a randomized way.

If we only use pure strategies, then we can create tables, showing again the strategic forms of games. Since the number of mixed strategies is almost never finite, strategic forms for games with mixed strategies will take a different form.

There is also the notion of a zero-sum two-player game with chance moves. The results concerning minimaxes and maximinis, saddle points and Nash equilibria remain true, because they rely only on the fact that Zermelo's algorithm can be used.

4.3.4. Theorem.

Let G be a zero-sum two player game with chance moves. Then G has a value v in the following sense: There are strategies σ_1 and σ_2 for the two players so that for each other pair of strategies τ_1 and τ_2 we have

$$\omega_1(\sigma_1, \tau_2) \geq v$$
$$\omega_2(\tau_1, \sigma_2) \geq -v$$

In words: The outcome *v* is the least the first player can expect, and the outcome –*v* is the least for the second player. Note that for zero-sum games we have again $\omega_1 = -\omega_2$, so that the pair of equation can also be written as

$$\omega_1(\sigma_1, \tau_2) \geq v$$
$$\omega_1(\tau_1, \sigma_2) \leq v$$