4. Games of Chance

4.1. Review of Basic Probability Theory

4.1.1. Definition.

A probability space is a pair (Ω, μ), where Ω is a finite set (called the sample space) and where μ is a function that assigns to every subset E of Ω a real number so that

1. $0 \le \mu(E) \le 1$ for each subset $E \subseteq \Omega$
2. $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$
3. $\mu(E \cup F) = \mu(E) + \mu(F)$ whenever $E \cap F = \varnothing$

Subsets of Ω are also called events.

The following properties are consequences of this definition:

4.1.2. Proposition.

If (Ω, μ) is a probability space, then for all subset $E, F \subseteq \Omega$ we have

1. $\mu(\Omega - E) = 1 - \mu(E)$
2. $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$

Proof. *Since the set $\Omega - E$ and E* are disjoint, the third property in the definition yields

$$\begin{aligned} 1 &= \mu(\Omega) \\ &= \mu\big((\Omega - E) \cup E\big) \\ &= \mu(\Omega - E) + \mu(E) \end{aligned}$$

This implies property (1).
Since the sets $F - E$ and E are disjoint, the third property of the definition gives

$$\begin{aligned} \mu(E \cup F) &= \mu\big(E \cup (F - E)\big) \\ &= \mu(E) + \mu(F - E) \end{aligned}$$

The sets $F - E$ and $E \cap F$ are also disjoint, hence

$$\begin{aligned} \mu(F) &= \mu\big((E \cap F) \cup (F - E)\big) \\ &= \mu(E \cap F) + \mu(F - E) \end{aligned}$$

Subtracting the second of those equations from the first gives

$$\mu(E \cup F) - \mu(F) = \mu(E) - \mu(E \cap F)$$

and this equation is equivalent to (2).

4.1.3. Definition.

Let (Ω, μ) *be a probability space. Two events* $E, F \subseteq \Omega$ *are called independent of each other, if they satisfy the equation* $\mu(E \cap F) = \mu(E)\mu(F)$

4.1.4. Example.

Let *A* be a finite set. We define

$$|A| = \text{number of elements of } A$$

If Ω is a fixed sample space, and if $E \subseteq \Omega$ is an event, then we define

$$\mu(E) = \frac{|E|}{|\Omega|}$$

In this way, (Ω, μ) becomes a probability space. In this probability space, two events are almost never independent.

If we are rolling a dye and would like to find the probability of outcomes, we could use the previous example with $\Omega = \{1,2,3,4,5,6\}$.

4.1.5. Example.

Two dice are rolled. In order to be able to distinguish between both dice, we assume that one dye is white and the other is black. Since there are 36 different possible outcomes, the sample space Ω^2 consists of all pairs *(i,j),* where both *i* and *j* represent numbers between 1 and 6:

$$\Omega^2 = \Omega \times \Omega \text{ where } \Omega = \{1,2,3,4,5,6\}$$

Since all of them are equally likely, we can define the outcome of an event *E* as in the previous example

$$\mu(E) = \frac{|E|}{|\Omega^2|} = \frac{|E|}{|\Omega|^2}$$

In this case, every outcome that has to do solely with the first (white) dye is independent with outcomes of the second (black) dye. For example, it is not hard to verify that the following two events are independent:

$$E = \{(i,j) : i \text{ is even}\}$$

$$F = \{(i,j) : j \text{ is less that } 5\}$$

As a matter of fact, the event $E \cap F$ represents all outcomes so that the white dye shows 2,4 or 6 and the black dye shows 1,2,3 or 4. Hence

$$E = \{2,4,6\} \times \{1,2,3,4,5,6\}$$
$$F = \{1,2,3,4,5,6\} \times \{1,2,3,4\}$$
$$E \cap F = \{2,4,6\} \times \{1,2,3,4\}$$
$$| E \cap F | = 12$$
$$\mu(E \cap F) = \frac{12}{36}$$
$$\mu(E) = \frac{18}{36} = \frac{3}{6}$$
$$\mu(F) = \frac{24}{36} = \frac{4}{6}$$

and therefore $\mu(E \cap F) = \mu(E)\mu(F)$

As we mentioned before, most of the time two events are not independent. In this case, we proceed as follows.

4.1.6. Definition.

If (Ω, μ) is a probability space, and if E and F are two events, then we define the conditional probability of F given E by

$$\mu(F \mid E) = \frac{\mu(E \cap F)}{\mu(E)}$$

In particular, we have

$$\mu(E \cap F) = \mu(E)\mu(F \mid E)$$

Interchanging the roles of E and F, we also obtain

$$\mu(E \cap F) = \mu(F)\mu(E \mid F)$$

The last two equations lead immediately to Bayes' Rule:

$$\mu(F \mid E)\mu(E) = \mu(E \mid F)\mu(F)$$

4.1.7. Definition.

Let (Ω, μ) be a probability space. A real valued function

$$f : \Omega \to \Re$$

is called a random variable. The expected value of a random variable is defined by

$$E(f) = \sum_{x \in \Omega} \mu(\{x\}) f(x)$$

There is a second expression for the expected value of a random variable, using the domain[2] of *f*.

4.1.8. Proposition.

Let (Ω, μ) *be a probability space, and let f be a random variable. Then*

$$E(f) = \sum_{r \in dom(f)} r\mu\left(f^{-1}(r)\right)$$

Proof. We first compute:

$$\begin{aligned}\sum_{x \in f^{-1}(r)} \mu(\{x\}) f(x) &= \sum_{x \in f^{-1}(r)} \mu(\{x\}) r \\ &= r \sum_{x \in f^{-1}(r)} \mu(\{x\}) \\ &= r\mu\left(f^{-1}(r)\right)\end{aligned}$$

This implies

$$\begin{aligned}E(f) &= \sum_{x \in \Omega} \mu(\{x\}) f(x) \\ &= \sum_{r \in dom(f)} \sum_{x \in f^{-1}(r)} \mu(\{x\}) f(x) \\ &= \sum_{r \in dom(f)} r\mu\left(f^{-1}(r)\right)\end{aligned}$$

The equation for the expected value in the preceding proposition is sometimes also written as

$$E(f) = \sum_{r \in dom(f)} r\mu(f = r)$$

We will try to avoid the last version.

What does the expected value measure? Suppose that a shaman wants to predict the future from the flight of birds. In order to do this, she needs to know the average speed of a bird. She knows that ¼ of all birds in her neighborhood are African swallows, ¼ of all birds are European swallows, 1/3 of all birds are sparrows and 5/12 of all birds are black birds – ops, something is wrong here: ¼ + ¼ + 1/3 + 5/12 = 15/12. That’s ¼ too many! Let’s count them again: ¼ of all birds are African swallows, ¼ of all birds are European swallows, 1/3 of all birds are sparrows, so only 1/6 need to be black birds. Yes, that’s better! Somehow, after counting the birds again, she still comes of with the first numbers, but let’s continue. The African swallows fly with a speed of 20 miles / hour, and so do the European swallows. The sparrows are much slower: they fly with 5 miles / hour, and the black bird fly with 8 miles / hour.

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[2] The domain is defined as $dom(f) = \{r : r = f(x) \text{ for some } x \in \Omega\}$. For each number *r* in the domain of *f*, we define $f^{-1}(r) = \{x \in \Omega : f(x) = r\}$

In this case, the sample space consists of all swallows – African or European, together with all sparrows and all black birds. The random variable *f* gives the speed of a particular bird. The domain of *f* consists of the numbers 20, 5 and 8. Furthermore, $f^{-1}(20)$ would be the set of all African and European swallows, and $\mu\left(f^{-1}(20)\right)$ would be the probability that a bird is an African or a European swallow – something like ¼ + ¼ = ½ or so. Therefore, in the average, the shaman expects a bird to fly with a speed of $\frac{1}{2}\times 20+\frac{1}{3}\times 5+\frac{1}{6}\times 8=10+3=13$ miles / hour. That's what the expected value gives her. Yeah! But she's still unhappy. Careful measurement gives an average speed of 10 miles / hour.

Forget about those birds!

4.1.9. Example.

Two dice are rolled. If the outcome has an even sum, you receive 12 points. If the outcome has an odd sum, you receive 15 points. How many points can you expect to receive in the average?

The probability space consist of all pairs *(x,y)*, where both *x* and *y* are between 1 and 6, and the random variable in question is given by

$$f(x,y)=\begin{cases}12 & \text{if } x+y \text{ is even}\\ 15 & \text{if } x+y \text{ is odd}\end{cases}$$

There are 36 different outcomes *(x,y)*. You will have an even sum, if either both dice show an even number (those are $3\times 3=9$ possibilities), or else both dice show an odd number (which are another 9 possibilities). So there are 18 possibilities that you receive 12 points, and also 18 possibilities that you receive 15. Hence the expected number of points is

$$E(f)=\frac{18}{36}\times 12+\frac{18}{16}\times 15=13.5$$

4.1.10. Example.

The Teutonic Lottery is played as follows: You buy a ticket for $1. This ticket has all the numbers between 1 and 49 printed in a table, and you chose 6 of those numbers. At the end of the week, each Saturday night the correct numbers are ceremoniously drawn from an urn. You win, depending on how many numbers you guessed correctly:

1. All six numbers guessed correctly: $1,000,000
2. Five numbers correct: $10,000
3. Four numbers correct: $100
4. Three numbers correct: $10

What are your chances to win? In the average, how much money do you expect to win in each given week? Would you play this lottery?

- There are $\binom{49}{6}$ different possibilities to pick 6 numbers from 49 numbers. Only 1 is correct, so there is 1 different possibility to guess all 6 numbers correctly.

If we only have 5 numbers correct, then we have $\binom{6}{5}$ many ways to pick 5 of the 6 numbers and $\binom{43}{1}$ to pick an incorrect number of the remaining 43 numbers. This gives a total of $\binom{6}{5}\binom{43}{1}$ possibilities to have exactly 5 numbers correct.

A similar argument shows that there are $\binom{6}{4}\binom{43}{2}$ ways to have exactly 4 numbers correct and $\binom{6}{3}\binom{43}{3}$ ways to have exactly 3 numbers correct.

So the probability to win \$ 1,000,000 is equal to $1/\binom{49}{6}$, the probability to win \$10,000 is equal to $\binom{6}{5}\binom{43}{1}/\binom{49}{6}$, the probability to win \$ 100 is equal to $\binom{6}{4}\binom{43}{2}/\binom{49}{6}$ and the probability to win \$ 10 is amounts to $\binom{6}{3}\binom{43}{3}/\binom{49}{6}$. Therefore you would expect to win

$$\frac{1}{\binom{49}{6}}\times \$\ 1{,}000{,}000+\frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times \$\ 10{,}000+\frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}}\times \$\ 100+\frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}}\times \$\ 10=\$\ 0.53$$

The people selling this lottery are keeping 47 cent of each dollar!