3.4. Saddle Points and Nash Equilibria

3.4.1. Definition.

Let G be a two-player zero-sum game. A pair of strategies (σ_1, σ_2) *for players 1 and 2, respectively, is called a saddle point for the game, if for each pair of strategies* (τ_1, τ_2) *for players 1 and 2, respectively, satisfies the inequalities*

$$\omega_1(\sigma_1, \tau_2) \leq \omega_1(\sigma_1, \sigma_2)$$

$$\omega_2(\tau_1, \sigma_2) \leq \omega_2(\sigma_1, \sigma_2)$$

The value

$$\omega(\sigma_1, \sigma_2) = (\omega_1(\sigma_1, \sigma_2), \omega_2(\sigma_1, \sigma_2))$$

is called a Nash Equilibirium of the game.

Since $\omega_1(\sigma_1, \sigma_2) = -\omega_2(\sigma_1, \sigma_2)$, the second equation (i.e. the equation $\omega_2(\tau_1, \sigma_2) \leq \omega_2(\sigma_1, \sigma_2)$) is equivalent to $\omega_1(\tau_1, \sigma_2) \geq \omega_1(\sigma_1, \sigma_2)$, and is frequently used in this form. In this case, only the number $\omega_1(\sigma_1, \sigma_2)$ is referred to as Nash Equilibrium

If *v* is the value of the game for the first player, then of course *v* is a Nash equilibrium and each pair of optimal strategies for the players is constitutes a saddle point for the game. In fact, as long as we talk about two-player zero-sum games, these are the only saddle points:

3.4.2. Theorem.

If G is a two-player zero-sum game, then the value of the game is the only Nash equilibrium, and the only saddle points can be obtained from a pair of optimal strategies for the two players.

Proof. Let (σ_1, σ_2) and (ξ_1, ξ_2) be two pairs of strategies for the two players, so that

1. (σ_1, σ_2) is a saddle point, and
2. (ξ_1, ξ_2) is a pair of optimal strategies for both players leading to the value *v* of the game:

$$\omega(\xi_1, \xi_2) = (v, -v)$$

Then $\omega_1(\sigma_1, \sigma_1)$ is a Nash equilibrium, and we would like to show that $\omega_1(\sigma_1, \sigma_1) = v$. The following identities are valid:

$\omega_1(\xi_1, \sigma_2) \geq v$ because ξ_1 is an optimal strategy for player 1
$\omega_2(\xi_1, \sigma_2) \geq \omega_2(\sigma_1, \sigma_2)$ because (σ_1, σ_2) is a saddle point
$\omega_1(\xi_1, \sigma_2) \leq \omega_1(\sigma_1, \sigma_2)$ multiplying the previous equation by -1
$\omega_1(\sigma_1, \sigma_2) \leq \omega_1(\sigma_1, \xi_2)$ because (σ_1, σ_2) is a saddle point
$\omega_2(\xi_1, \xi_2) \leq \omega_2(\sigma_1, \xi_2)$ because ξ_2 is an optimal strategy for player 2
$\omega_1(\sigma_1, \xi_2) \leq \omega_1(\xi_1, \xi_2)$ multiplyint the previous equation by -1
$\omega_1(\sigma_1, \xi_2) \leq v$

It follows that $v \leq \omega_1(\xi_1, \sigma_2) \leq \omega_1(\sigma_1, \sigma_2) \leq \omega_1(\sigma_1, \xi_2) \leq v$ and hence $\omega_1(\sigma_1, \sigma_1) = v$.