3.3. Finding Value from the Strategic Form of a Game

We will now develop a method to compute the value of a two-player zero-sum from the strategic form of the game.

3.3.1. Definition.

Let G be an n – player games. If for each number i between 1 and n σ_i for player i and 2, respectively, and if

$\omega(\sigma_1,\cdots,\sigma_n)=(r_1,\cdots,r_n)$, *then we define*

$$\omega_j(\sigma_1,\cdots\sigma_n)=r_j$$

Hence ω_j is the *j*th coordinate function of ω:

$$\omega(\sigma_1,\cdots,\sigma_n)=(\omega_1(\sigma_1,\cdots,\sigma_n),\omega_2(\sigma_1,\cdots,\sigma_n),\cdots\omega_n(\sigma_1,\cdots,\sigma_n))$$

In particular, if we are dealing with a two-player game, then

$$\omega(\sigma,\tau)=(\omega_1(\sigma,\tau),\omega_2(\sigma,\tau))$$

If *G* is a two-player zero-sum game, then

$$\omega_1(\sigma,\tau)=-\omega_2(\sigma,\tau)$$

3.3.2. Proposition.

Let If σ_1,σ_2 are optimal strategies for players 1 and 2, respectively, and if v is the value of the game, the for each pair of strategies τ_1,τ_2 of players 1 and 2, respectively, we have

$$\omega_1(\sigma_1,\tau_2)\ge v$$

$$\omega_2(\tau_1,\sigma_2)\ge -v$$

and

$$\omega_2(\sigma_1,\tau_2)\le -v$$

$$\omega_1(\tau_1,\sigma_2)\le v$$

Proof. From the definitions, we know that

$$\omega(\sigma_1,\tau_2) = (r,-r) \text{ with } r \geq v$$
$$\omega(\tau_1,\sigma_2) = (s,-s) \text{ with } -s \geq -v$$

This implies

$$\omega_1(\sigma_1,\tau_2) \geq v$$
$$\omega_2(\tau_1,\sigma_2) \geq -v$$

and, since $\omega_1(\sigma,\tau) = -\omega_2(\sigma,\tau)$

$$\omega_2(\sigma_1,\tau_2) \leq -v$$
$$\omega_1(\tau_1,\sigma_2) \leq v$$

The previous identities will be used in the following proof:

3.3.3. Theorem.

The value v of a two-player zero-sum game G can be computed from each of the following four formulas:

$$v = \max\left\{\min\left\{\omega_1(\xi,\eta) : \eta \text{ is a strategy of player 2}\right\} : \xi \text{ is a strategy of player 1}\right\}$$
$$-v = \max\left\{\min\left\{\omega_2(\xi,\eta) : \xi \text{ is a strategy of player 1}\right\} : \eta \text{ is a strategy of player 2}\right\}$$

and

$$v = \min\left\{\max\left\{\omega_1(\xi,\eta) : \xi \text{ is a strategy of player 1}\right\} : \eta \text{ is a strategy of player 2}\right\}$$
$$-v = \min\left\{\max\left\{\omega_2(\xi,\eta) : \eta \text{ is a strategy of player 2}\right\} : \xi \text{ is a strategy of player 1}\right\}$$

Proof. Let v be the value of the game G. Then there are strategies σ_1 for player 1 and σ_2 for player 2, so that for all possible strategies τ_1 for player 1 and τ_2 of player 2 we have

$$\omega_1(\sigma_1,\tau_2) \geq v$$
$$\omega_2(\tau_1,\sigma_2) \geq -v$$

and

$$\omega_2(\sigma_1,\tau_2) \leq -v$$
$$\omega_1(\tau_1,\sigma_2) \leq v$$

Hence

$$\min\{\omega_1(\sigma_1,\eta):\eta \text{ is a strategy for player 2}\} \geq v$$

and therefore

$$\max\{\min\{\omega_1(\xi,\eta):\eta \text{ is a strategy of player 2}\}:\xi \text{ is a strategy of player 1}\} \geq v$$

Also, since $\omega_1(\tau_1,\sigma_2) \leq v$ for each choice of τ_1, the inequality

$$\min\{\omega_1(\xi,\eta):\eta \text{ is a strategy of player 2}\} \leq \omega_1(\xi,\sigma_2) \leq v$$

holds for each value of ξ. This implies the first of our four equation claimed in the statement of the theorem:

$$\max\{\min\{\omega_1(\xi,\eta):\eta \text{ is a strategy of player 2}\}:\xi \text{ is a strategy of player 1}\} = v$$

The second of those four equations can be verified in the same way after exchanging the roles of players 1 and 2. The third equation follows from the second equation: Multiplying the equation

$$-v = \max\{\min\{\omega_2(\xi,\eta):\xi \text{ is a strategy of player 1}\}:\eta \text{ is a strategy of player 2}\}$$

by *-1* gives

$$\begin{aligned}
v &= -\max\{\min\{\omega_2(\xi,\eta):\xi \text{ is a strategy of player 1}\}:\eta \text{ is a strategy of player 2}\} \\
&= \min\{-\min\{\omega_2(\xi,\eta):\xi \text{ is a strategy of player 1}\}:\eta \text{ is a strategy of player 2}\} \\
&= \min\{\max\{-\omega_2(\xi,\eta):\xi \text{ is a strategy of player 1}\}:\eta \text{ is a strategy of player 2}\} \\
&= \min\{\max\{\omega_1(\xi,\eta):\xi \text{ is a strategy of player 1}\}:\eta \text{ is a strategy of player 2}\}
\end{aligned}$$

The proof of the last equation is similar.

The results of the previous theorem come with some fancy words: If *v* is the value of the game, then the optimal strategies for both players result in a row and a column of the matrix at which the value *v* can be found. All other entries in this row will be larger than or equal to *v*, and all other values in that column will be less than or equal to *v*. In other words, we have found a *saddle point* of the matrix (table) representing the game.

Also, because neither of the players can improve the outcome of the game unless the other player makes a mistake, both players should have found true happiness. They are doing as well as they possibly can. They have found inner balance and peace – they are at a *Nash equilibirium*. We will further discuss these ideas in a subsequent section.

In the table of the strategic form, the rows represent the strategies of the first player, and the columns represent strategies of the second player. In every row we take the minimum value of the first coordinates. This minimum value would be the least number of points the first player can expect if she uses the strategy given by the current row. Then we take the maximum of all those minimum values. This number *v* would be the value of the game, and each row for which this maximum is obtained would yield an optimal strategy for the first player.

We also could make a corresponding computation for the second player: Take the minimum value of the second coordinates of each column, and then find the maximum value of all those minima. Each column for which this maximum is obtained belongs to an optimal strategy for the second player.

Many times the method for the second player is carried out differently. Since for zero-sum games the first coordinate function ω_1 of the function ω determines the second coordinate via the equation $\omega_1 = -\omega_2$, only the first coordinate function is needed. Therefore, we need only to enter the function ω_1 into the tables of the strategic form of the game. For the function ω_1 we have to following interesting corollary:

3.3.4. Corollary.

Let G be a zero-sum two-player game. If ω_1 is the outcome function for the first player, then

$$\max\left\{\min\left\{\omega_1(\xi,\eta) : \eta \text{ is a strategy for player 2}\right\} : \xi \text{ is a strategy for player 2}\right\}$$
$$= \min\left\{\max\left\{\omega_1(\xi,\eta) : \xi \text{ is a strategy for player 1}\right\} : \eta \text{ is a strategy for player 1}\right\}$$

So the optimal strategy for the second player and the value of the game can be found in the following way: For each column find the maximal entry. Of all those maximum entries find the minimum *v*. This minimum is the value of the game, and each column in which this minimum occurs represents an optimal strategy for the second player.

Suppose that the following table gives the strategic form of a game.

	Strategy a	Strategy b	Strategy c	Strategy d
Strategy 1	(10,-10)	(10,-10)	(3,-3)	(3,-3)
Strategy 2	(9,-9)	(2,-2)	(9,-2)	(2,-2)

Since this is a zero-sum game, the second coordinates of the outcomes do not have to be mentioned

	Strategy a	Strategy b	Strategy c	Strategy d
Strategy 1	10	10	3	3
Strategy 2	9	2	9	2

The minimum of the first row is 3 and the minimum of the second row is 2. The maximum of 3 and 2 is 3, and therefore

1. The value of the game is 3
2. It is optimal for player 1 to follow strategy 1.

Using columns to find the optimal strategy for the second player, we find that the maximum value for the columns are 10,10, 9, 3, respectively. The minimum of those four values is 3. Hence

1. The value of the game is 3.
2. It is optimal for the second player to follow Strategy d.

Notice that indeed, as predicted by the theory, the maximum of the minima over the rows equals the minimum of the maxima over the columns. However, what is wrong with the following example?

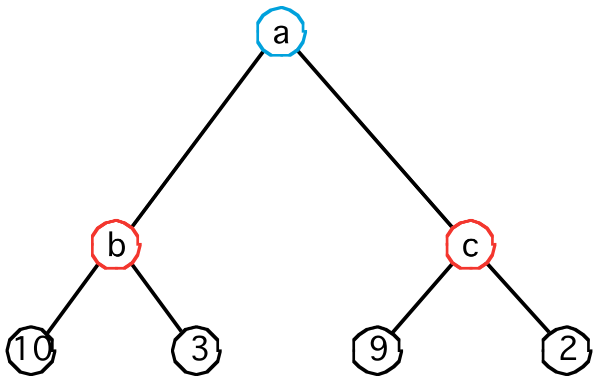
	4	1
	2	3

The minima over the rows are 1and 2, and 2 is the larger of the numbers. So the maximum over the minima of the rows is 2. The minimum over the maxima of the columns is different, namely 3.

It should be noted that each player could have more than one strategy. For examples, this would be the case if all the outcomes of the game would be identical, now matter how the players are moving.

Also, for each player it might not be a good idea to be lazy. Just by looking at the optimal strategies for the first player, the second player might think it is a good idea just to block the optimal strategies of the first player. Surely the first player will use one of his optimal strategies. But all the sudden, the first player sees what the second player is doing consistently and changes strategies. Since the second player is not prepared to block other strategies, she now looses badly.

The following example illustrates this.



The value of the game is 3, the optimal strategy for the blue player is to move from a to b. If the red player just wants to defend against this strategy, she would decide to move from b to the 3 and from c – now she doesn't care anymore. This will surely never happen. So she just marks to be move from c to 9 in her strategy. The blue player – looking into her cards – decides now to move from a to c and wins 9 points instead of only 3.