3.2. Values of Games and Zermelo's Algorithm

3.2.1. Definition.

Consider a zero-sum two-player game G. A real number v is called a value of G, if there are strategies σ_1 *for player 1 and* σ_2 *for player 2, so that for all possible strategies* τ_1 *for player 1 and* τ_2 *of player 2 we have*

$$\omega(\sigma_1, \tau_2) = (r, -r) \text{ with } r \geq v$$

$$\omega(\tau_1, \sigma_2) = (s, -s) \text{ with } -s \geq -v$$

In other words, the first player can guarantee to win at least *v* of the available points, and second player has a way to ensure that he wins at least –*v* points. Since the number *v* could be less than zero, the negative sign does not mean that the first player wins.

3.2.2. Proposition.

Let v be a value of a zero-sum two-player game. Then there are strategies σ_1 *for player 1 and* σ_2 *for player 2 so that*

$$\omega(\sigma_1, \sigma_2) = (v, -v)$$

Proof. Let σ_1 and σ_2 be the strategies from the definition of a value of a game. Then there is a number *t* so that

$$\omega(\sigma_1, \sigma_2) = (t, -t)$$

We now can apply the definition with $\tau_1 = \sigma_1$ and $\tau_2 = \sigma_2$ and obtain

$$\omega(\sigma_1, \sigma_2) = (r, -r) \text{ with } r \geq v$$

$$\omega(\sigma_1, \sigma_2) = (s, -s) \text{ with } -s \geq -v$$

Of course, this implies that $r = s = t$ and therefore

$$t \geq v$$

$$-t \geq -v$$

and hence $t = v$.

Next, we verify that a game has a most one value:

3.2.3. Proposition.

Let v_1 *and* v_2 *be two values of a zero-sum two-player game. Then*

$$v_1 = v_2$$

Proof. We begin by picking strategies σ_1, σ_2, γ_1 and γ_2 satisfying the properties of the definition of a value. This means that for all possible strategies τ_1 for player 1 and τ_2 of player 2 we have

$$\omega(\sigma_1,\tau_2)=(r_1,-r_1) \text{ with } r_1 \geq v_1$$
$$\omega(\tau_1,\sigma_2)=(s_1,-s_1) \text{ with } -s_1 \geq -v_1$$

and

$$\omega(\gamma_1,\tau_2)=(r_2,-r_2) \text{ with } r_2 \geq v_2$$
$$\omega(\tau_1,\gamma_2)=(s_2,-s_2) \text{ with } -s_2 \geq -v_2$$

If we pick $\tau_1=\sigma_1$ and $\tau_2=\gamma_2$, then the first equation of the first pair of equation yields

$$\omega(\sigma_1,\gamma_2)=(x,-x) \text{ with } x \geq v_1$$

while the second equation of the second pair of equation gives

$$\omega(\sigma_1,\gamma_2)=(x,-x) \text{ with } -x \geq -v_2$$

We conclude that $v_1 \leq v_2$. If we use the second equation of the first pair and the first equation of the second pair of equation, then we obtain $v_2 \leq v_1$. Hence $v_1 = v_2$.

We now would like to show that every two-player zero-sum game has indeed a value. We will use the recursive definition of trees to do so. We also need the idea of subgames. A subgame of a game keeps the labeling of the nodes (using 1 and 2 the two players) and the leaves (using pairs *(s, t)* describing the outcomes of the game), and simply starts "later in the game". Again, this definition is best done recursively:

3.2.4. Definition.

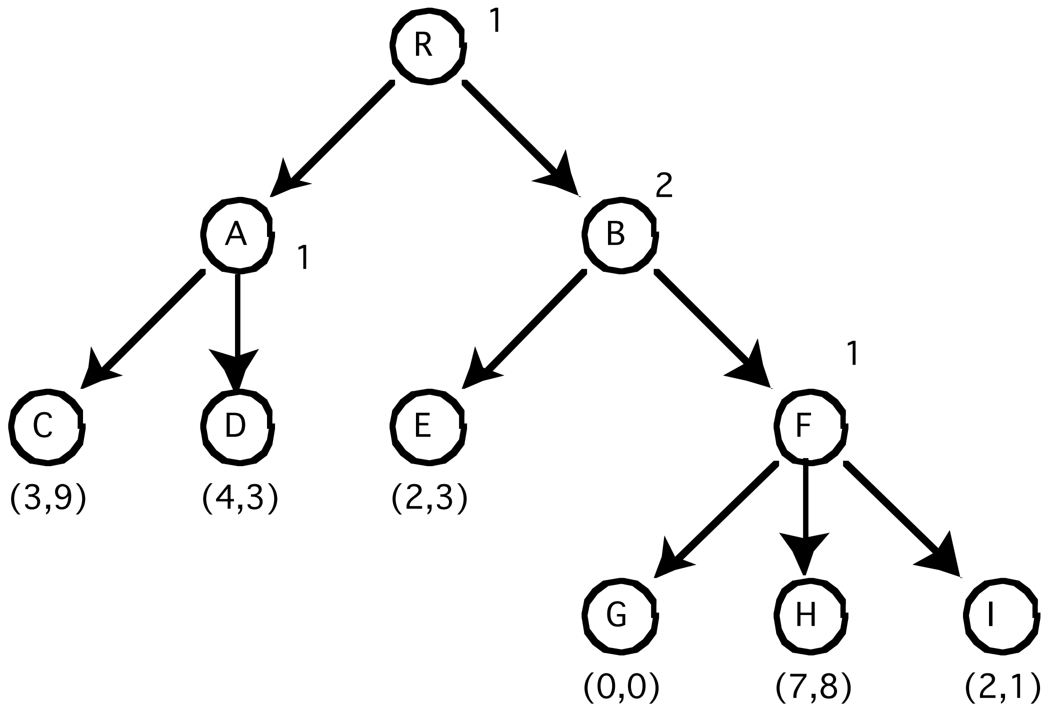
Let G be a game with game tree t, player function p and outcome function ω.

1. *If t consists one leaf, then G is the only subgame of G.*
2. *If* $t=[t_1\oplus\cdots\oplus t_n]$, *then the subgames of G are*

a. *G itself.*

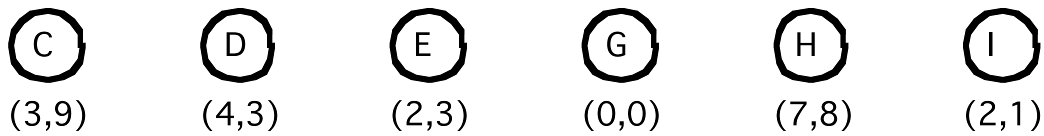
b. *All subgames of the games* G_i, *where* G_i *has game tree* t_i *and where the player function of* G_i *are the functions p and* ω, *restricted to the nodes and leaf of* t_i.

If we use the following game:

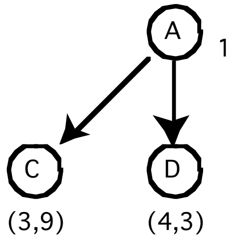


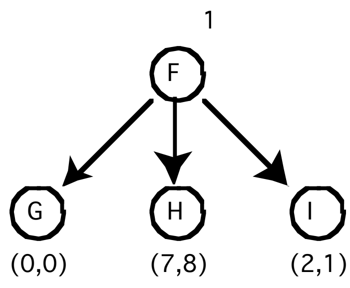
then a list of all subgames looks as follows:

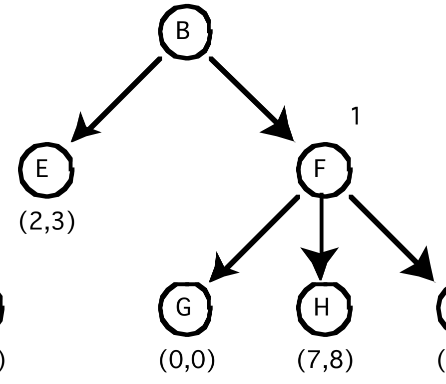
Subgames with one node:



Subgames with more than one node other than *G* itself:







3.2.5. Theorem.

Every zero-sum two-player game G with a finite game tree t has a value.

Proof. We prove this theorem by induction on the number of nodes of the game tree *t*. If *t* has only one node, then this node is a leaf. If $(r,-r)$ is the label at that leaf, no matter which strategy the players are using, the outcome of the game will have the value $(r,-r)$, and therefore the value of the game is *r*.

Assume that we have verified the theorem for all games that have game trees with fewer than *n* node, and that we would like to prove the theorem for a game *G* whose game tree *t* has exactly *n* nodes.

We have two cases to consider, depending on whether player 1 or 2 moves first in the game. Both cases are analogous – we simply would have to change the names of the players and the coordinates in the outcome function ω. Hence we may assume without loss of generality that player 1 moves first.

By the recursive definition of trees,

$$t = [t_1 \oplus \cdots \oplus t_k]$$

Let G_j be the subgame of *G* with game tree t_j. Then, since at least the root of *t* is missing from t_j, the game tree of G_j has fewer than *n* nodes and therefore a value v_j We let

$$v = \max\{v_1, \ldots v_k\}$$

We will show that *v* is a value for the game *G*.

There are strategies $\sigma_{1,j}$ and $\sigma_{2,j}$ for players 1 and 2, respectively, that fulfill the definition of values for the subgame G_j .We use those strategies to define strategies σ_1 and σ_2 for players 1 and 2 that are optimal for the game *G*, as follows:

If *R* is the root of *t*, and if j_0 is a fixed index so that $v = v_{j_0}$, then

$$\sigma_1(R) = R_{j_0}$$

If *N* is a node of *t* that is not the root and not a leaf, and it is the first player's turn to move at this node, then

$$\sigma_1(N) = \sigma_{1,j}(N) \text{ if } N \text{ is a node of the subtree } t_j$$

If *N* is a node of *t* that is not a leaf, and the second player moves at this node, then

$$\sigma_2(N) = \sigma_{2,j}(N) \text{ if } N \text{ is a node of the subtree } t_j$$

Now let τ_1 and τ_2 be strategies for the game *G* for players 1 and 2, respectively. Restricting those strategies to nodes of the subtree t_j

leads to strategies $\tau_{1,j}$ and $\tau_{2,j}$ for the subgame G_j. Moreover, if $\tau_1(R)$ is the root of the subtree t_j, then the outcome of the game is obtained by following the first player from the root *R* to the root of t_j, and then following the strategies $\tau_{1,j}$ and $\tau_{2,j}$, once we are inside the tree:

$$\omega(\tau_1,\tau_2)=\omega(\tau_{1,j},\tau_{2,j})$$

Using the strategy σ_1, the first player moves from the root *R* to the root of the subtree t_{j_0}. This, together with the fact that σ_{1,j_0} is a strategy guaranteeing the first player an outcome of at least v_{j_0}, leads to the computation

$$\begin{aligned}\omega(\sigma_1,\tau_2) &= \omega(\sigma_{1,j_0},\tau_{2,j_0})\\ &= (r,-r) \text{ with } r \geq v_{j_0}\\ &= (r,-r) \text{ with } r \geq v\end{aligned}$$

We now use the optimal strategy σ_2 for the second player and an arbitrary strategy τ_1 for the first player. If the first player's strategy τ_1 moves from the root *R* to the root of the subtree t_j, then

$$\begin{aligned}\omega(\tau_1,\sigma_2) &= \omega(\tau_{1,j},\sigma_{2,j})\\ &= (r,-r) \text{ with } -r \geq -v_j\\ &= (r,-r) \text{ with } r \leq v_j\\ &= (r,-r) \text{ with } r \leq \max\{v_1,\cdots v_k\}\\ &= (r,-r) \text{ with } r \leq v\end{aligned}$$

Hence *v* is a value for the game *G*.

The last theorem shows that every finite game has a value and therefore also an optimal strategy. For many games, including Hex, the strategy is not known. For other interesting games, like chess, not even the value is known.

3.2.6. Zermelo's Algorithm.

Zermelo's algorithm picks a strategy for the players in the following way: If player 1is at a node *N*, he will look at all subgames starting at the children of the node *N* and find their values. He then will move in the direction of the highest value. The second player will move in the direction of the smallest value.