1. (a) Let $b(1) = \bar{b}$. For $\widehat{b}_1 \leq \bar{b}$, $pr(b(v_1) \leq \widehat{b}_1) = pr(v_1 \leq b^{-1}(\widehat{b}_1)) = b^{-1}(\widehat{b}_1)$.

   (b) $\frac{d}{dv_1}\Pi(v_1) = \frac{\partial}{\partial v_1}\pi(v_1, b_1) + \frac{\partial}{\partial b_1}\pi(v_1, b_1)\frac{\partial b_1}{\partial v_1} = \frac{\partial}{\partial v_1}\pi(v_1, b_1) = b^{-1}(b_1(v_1))$.

   (c) $\frac{d}{dv_1}\Pi(v_1) = v_1$. $\Pi(v_1) = \int_0^{v_1} x dx = \frac{v_1^2}{2}$. Since $\Pi(v_1) = (v_1 - b(v_1))b^{-1}(b(v_1))$, $\frac{v_1^2}{2} = (v_1 - b(v_1))v_1$. From this, $b(v_1) = \frac{v_1}{2}$.

   (d) For $\widehat{b}_1 \in [0, \frac{1}{2}], b^{-1}(\widehat{b}_1) = 2\widehat{b}_1$.Note that $(v_1 - \widehat{b}_1)2\widehat{b}_1$ is a concave function. $\frac{\partial}{\partial \widehat{b}_1}(v_1 - \widehat{b}_1)2\widehat{b}_1 = (v_1 - \widehat{b}_1)2 - 2\widehat{b}_1 = 0$. From this, $\widehat{b}_1 = \frac{v_1}{2}$.

   (e) The bidders choose $(\widehat{v}_1, \widehat{v}_2)$ from $[0,1] \times [0,1]$. $\left(\frac{\widehat{v}_1}{2}, \frac{\widehat{v}_2}{2}\right)$. if $\frac{\widehat{v}_i}{2} > \frac{\widehat{v}_j}{2}$, $j \neq i$, bidder i wins the object and pays $\frac{\widehat{v}_i}{2}$ and bidder j pays nothing. If $\frac{\widehat{v}_i}{2} = \frac{\widehat{v}_j}{2}$, then the winner is chosen randomly with equal probability. (the latter actually does not matter since it is a probability zero event.)

   (f) Let $v_1 < \widehat{b}_1$. If $\widehat{b}_1$ does not win, v_1 does not win either. If $\widehat{b}_1$ wins and the second price is greater than v_1, bidder 1 would have been better off with bid of v_1. If the second price was smaller than v_1, v_1 would have won and paid the same second price. If the second price is equal to v_1, bidder 1 gets zero surplus by bidding $\widehat{b}_1$. Bidder 1 also gets zero surplus by bidding v_1 whether he wins or not. The case of $v_1 > \widehat{b}_1$ is similar.

   (g) $\int_0^{v_1}(v_1 - v_2)dv_2 = v_1 * v_1 - \frac{v_1^2}{2} = \frac{v_1^2}{2}$. This is the same as $\Pi(v_1)$ computed in c.

2. (a) $x_1 = \frac{1}{2}, y_1 = \frac{1}{2p}, x_2 = \frac{1}{2} + \frac{1}{2}p, y_2 = \frac{1}{2}\frac{1+p}{p}$

   (b) For $p > \frac{1}{9}$, no profit maximizing production vector, $p = \frac{1}{9}$, any non-negative input bundle of (x, y), $x = y$ with output $10y$ will result in zero profit. For $p < \frac{1}{9}$, zero input and zero output maximizes profit.

   (c) $x_1 = \frac{1}{2}, y_1 = \frac{9}{2}, x_2 = \frac{1}{2} + \frac{1}{2} * \frac{1}{9} = \frac{5}{9}, y_2 = \frac{1}{2}\frac{1+\frac{1}{9}}{\frac{1}{9}} = 5$. The production allocation are: inputs: $\frac{1}{2} + \frac{4}{9} = \frac{17}{18}$. So, inputs of labor and rice are $(\frac{17}{18}, \frac{17}{18})$ and has gross rice output of $\frac{170}{18}$. Net output are $\frac{17}{2}$. So total supply of rice is $\frac{17}{2} + 1 = \frac{19}{2}$.

   (d) $p = 2$. $x_2 = 1, y_2 = 1$.

   (e) $x_1 = \frac{1}{2}, y_1 = \frac{1}{4}$, input $(\frac{1}{2}, \frac{1}{2})$, output $\frac{3}{4}$, net output $\frac{1}{4}$.

   (f) yes, the first welfare theorem still applies.

3. (a) A pure strategy for a consumer is a function assigning to each price pair (m, p) a decision whether or not to go to the theater, and how much popcorn to buy. The function can be represented as $f : \mathbb{R}^2 \to \{0, 1\} \times \mathbb{R}_+$, assigning to each (m, p) a vector $f(m, p) = (a, q)$, where $a = 1$ if the consumer goes to the theater and $a = 0$ otherwise, and where q is the amount of popcorn bought, with $q = 0$ when $a = 0$. An example is the function $f(m, p) = (0, 0), \forall (m, p) \in \mathbb{R}^2$, interpreted as meaning that the consumer does not go to the theater no matter what the prices are.

   (b) The economic idea is that maximum total surplus can be attained. The theater (principal) charges prices (offers a contract) that maximizes its payoff subject to a participation constraint for each consumer (agent). Omitting the fixed cost, K, the total surplus attainable from each consumer is $5 + [5q - (q^2/2)] - q$. Its maximum value is 13, attained at $q = 4$. An example of SPE is with price equal to marginal cost: $p = 1$ and admission equal to the maximum surplus from a single consumer: $m = 13$. Each consumer chooses $a = 1$ and $q = 5 - p$ whenever (m, p) is such that a nonnegative payoff is attainable, i.e., whenever $m \leq 5 + [5q - (q^2/2)] - pq$ for some q. Otherwise the consumer chooses $(a, q) = (0, 0)$. (Note that in specifying an SPE, we must specify the consumer's entire strategy.) In this SPE, each consumer sees the movie, buys 4 boxes of popcorn, and gets 0 payoff. No consumer can get a higher payoff given the prices. The theater gets a profit of \$13 from each consumer and a payoff $13N - 1 > 0$ equal to the total surplus. Thus no higher payoff is possible for it. The outcome is Pareto efficient. It also maximizes total surplus and is best from the point of view of the theater.

   (c) Variation in K only changes the theater's payoff by a constant. It has no effect on the set of optimal consumer strategies and no effect on the theater's pricing, hence no effect on SPE, unless $K \geq 13N$. If $K = 13N$, then every equilibrium from the case $K < 13N$ remains an equilibrium. There are also equilibria in which the theater offers prices that all consumers reject and no one sees the movie. If $K > 13N$ then it is not possible for the theater to attain a nonnegative payoff by showing the movie, so in every equilibrium, no consumer sees the movie.

(d) No. There are only N consumers who are willing to pay anything for popcorn. The maximum profit the theater can get by selling popcorn to a consumer who does not pay for admission is \$4, with $p = 3$. The theater makes more profit when the consumer is admitted, as in the SPE of part b, so it should charge prices that induce the consumers to pay both for admission and for popcorn.

(e) For a near 1, the theater wants to get the largest surplus from the original type of consumer, so it charges $m = 13$ and $p = 1$ as before and gets profit $13aN - 1$. For a near 0, the theater wants to attract the new type of consumer, so it charges $m = 3$. Then the most it can get from the original type of consumer is the \$3 admission fee plus the monopoly profit of \$4, which it gets by charging $p = 3$. The original consumer type gets payoff 4 and the theater's payoff is $3N+4aN-1$. In both cases, the original consumers' strategies are as in part b, and the new consumers buy tickets if and only if $m \leq 3$. The first outcome occurs in equilibrium in case $13aN > 3N + 4aN$, i.e., when $a > 1/3$. The second outcome occurs in case $a < 1/3$ and either can occur if $a = 1/3$. In either case, the outcome is less efficient than with perfect information. In the first outcome, consumers who value the movie miss it, though it costs nothing to allow them to see it. In the second outcome, consumers consume inefficiently little popcorn since the price is above marginal cost. In some cases, the outcome is constrained inefficient since, as shown in part f, it may be possible for the theater to get a higher payoff without reducing the consumers' payoffs.

(f) he theater can offer two contracts designed for the different consumer types. Let m be a fee for admission without popcorn and f be a fee that pays for admission plus q boxes of popcorn. The theater can choose (m, f, q) so that every consumer accepts a contract, and the contracts maximize $(1 - a)mN + afN - aqN - 1$ subject to $3 - m \geq 0$, $5 + 5q - (q^2/2) - f \geq 0$ and $5 + 5q - (q^2/2) - f \geq 5 - m$. The first order conditions imply that at a solution, $m = 3$, $q = 4$, $f = 5q - (q^2/2) + m = 15$. The theater's payoff is $3N + 8aN - 1$, which equals the payoff from an optimal direct mechanism for the theater in which each type of consumer is willing to report its true type and every consumer is admitted to the movie. By the revelation principle, the theater cannot do better when every consumer is admitted. With these contracts, the theater's payoff is higher than in part e if $a < 3/5$. Each type of consumer gets a nonnegative payoff, so when $1/3 < a < 3/5$, the outcome in part e is constrained inefficient. With the same information, the theater can get a higher payoff than in part e without hurting any consumer type. The strategy is equivalent to charging \$3 for admission and offering admitted consumers the chance to buy 4 boxes of popcorn for \$12. (A strategy with the same outcome would be to charge \$3 for admission, then charge \$8 for the right to buy popcorn at \$1 per box.) Consumers who value popcorn are indifferent between buying 4 boxes and not buying any. In equilibrium with the strategies above, they do buy popcorn. If $a < 1/3$, then the original consumer type is worse off with the contracts in part f than in part e.

(g) When $a < 1/3$, the price of popcorn is high in the equilibrium in part e because consumers who value popcorn also value the movie highly. If there is no such correlation between value attached to the movie and value attached to using the bathroom, there is no reason for charging a price above marginal cost for the bathroom. Even if granting access to the bathroom has positive marginal cost for the theater, the optimal fee for use could be 0 if there is a small transaction cost of collecting a fee.

4. (a) Maximized at $u_1 = u_2 = u_3$.

(b) $px_i + m_i = M_i$

(c) Demands and indirect utility: $x_1 = \frac{M_1}{2}, m_1 = \frac{M_1}{2}, v_1 = \frac{M_1}{2}; x_2 = \frac{4M_2}{5}, m_2 = \frac{M_2}{5}, v_2 = \sqrt{5M_2}$; $x_3 = \frac{M_3}{5}, m_3 = \frac{4M_3}{5}; v_3 = \sqrt{5M_3}$. Using $v_1 = v_2 = v_3$ and $M_1 + M_2 + M_3 = \bar{M}$, $M^* = (30, 45, 45)$. Unfair in terms of income distribution, fair in terms of utility distribution.

(d) For $\bar{M} = 60$, $M^* = (20, 20, 20)$.

(e) For expenditure functions, $E_1(v_1^*) + E_2(v_2^*) + E_3(v_3^*) = \bar{M}$.

(f) $v_1(40) = 20, v_2(40) = v_2(40) = 10\sqrt{2} \to W = 10\sqrt{2} \approx 14.14$ vs.$W^* = 15$.

(g) $(-5, 10, -5)$

(h) The feasible tax schemes are $(T, T, -2T)$ for $T \in \mathbb{R}$, resulting in the income distributions $(35 + T, 35 + T, 50 - 2T)$. For these, $v_1 = \frac{35+T}{2}$, $v_2 = \sqrt{5(35 + T)}$, $v_3 = \sqrt{5(50 - 2T)}$). The second-best $T = 5$, which results in the income distribution $(40, 40, 40)$ and $W = v_2 = v_3 \approx 14.14$, as in part f.

(i) Voucher vs. money will lower 3's utility and W from the first-best. Since it would lower W, the motivation to increase x_3 must derive from a source other than maximizing W, as specified.