

Microeconomics

Question 1.1

A function $u(x_i)$ is strictly concave if $\forall x_i, x'_i$, and $\forall \lambda \in (0, 1)$,

$$u(\lambda x_i + (1 - \lambda)x'_i) > \lambda u(x_i) + (1 - \lambda)u(x'_i)$$

To show that the differentiable function $u(x_i) = 10x_i - x_i^2$ is strictly concave, you can either use this definition direction or use the implication that a differentiable function with a strictly negative second derivative is strictly concave. In this case, the second derivative is $u''(x_i) = -2 < 0$. The workers are risk averse since, for any lottery with expected value EV , as $u(EV) > EU$.

Question 1.2

The expected wage is:

$$\frac{2}{3}(1) + \frac{1}{3}(4) = 2$$

The expected utility is:

$$U(1, 4) = \frac{2}{3}(10(1) - 1^2) + \frac{1}{3}(10(4) - 4^2) = 14$$

Question 1.3

This makes the workers better off as the utility $U(2, 2) > EU$. The intuition is that the workers are risk averse, and being paid a fixed wage $\bar{x} = 2$ regardless of the state of the world eliminates all of the for the workers without reducing the expected payment.

The lowest wage the firm can pay, \underline{x} , is given by the equation:

$$10\underline{x} - \underline{x}^2 = 14 = U(1, 4)$$

Question 1.4

In light of the answers to [1.1] - [1.3], firms can take advantage of the risk averse preferences of the workers by hiring them at a lower wage \underline{x} but insuring them against the bad states. But this requires that wages not be reduced during a recession.

Question 2.1

We want to solve the following utility maximization problem

$$\max_{x_1, x_2} p(\ln x_1 + \ln x_2) - p_1 x_1 - p_2 x_2 \quad (1)$$

First order conditions are

$$\frac{\partial \pi}{\partial x_1} = p \frac{1}{x_1} - p_1 = 0 \quad (2)$$

$$\frac{\partial \pi}{\partial x_2} = p \frac{1}{x_2} - p_2 = 0 \quad (3)$$

Second order conditions are

$$\pi_{11} = -\frac{p}{x_1^2} < 0 \quad (4)$$

And

$$\begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -\frac{p}{x_1^2} & 0 \\ 0 & -\frac{p}{x_2^2} \end{vmatrix} = \frac{p^2}{x_1^2 x_2^2} > 0$$

From equation (1), we can find demand for good 1 as

$$x_1 = \frac{p}{p_1} \quad (5)$$

Question 2.2

$$\begin{aligned} \min_{x_1, x_2} p x_1 + p x_2 & \quad (6) \\ \text{s.t. } f(x_1, x_2) = \ln x_1 + \ln x_2 = q & \end{aligned}$$

Let

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda(q - \ln x_1 - \ln x_2)$$

First order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda \left(\frac{1}{x_1} \right) = 0 \quad (7)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda \left(\frac{1}{x_2} \right) = 0 \quad (8)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q - \ln x_1 - \ln x_2 = 0 \quad (9)$$

We have $C(q) = p_1 x_1 + p_2 x_2$ Therefore if we differentiate C w.r.t q (which is the MC), we get

$$\frac{\partial C}{\partial q} = p_1 \frac{\partial x_1}{\partial q} + p_2 \frac{\partial x_2}{\partial q} \quad (10)$$

From the first order conditions, we get

$$p_1 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} \quad (11)$$

$$p_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} \quad (12)$$

Substitute (11) (12) into (10), we get

$$\begin{aligned} \frac{\partial C}{\partial q} &= \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} \frac{\partial x_1}{\partial q} + \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} \frac{\partial x_2}{\partial q} \\ &= \lambda \left(\frac{\partial f(x_1, x_2)}{\partial x_1} \frac{\partial x_1}{\partial q} + \frac{\partial f(x_1, x_2)}{\partial x_2} \frac{\partial x_2}{\partial q} \right) \end{aligned} \quad (13)$$

Since $f(x_1, x_2) = q$, therefore $\frac{\partial f(x_1, x_2)}{\partial x_1} \frac{\partial x_1}{\partial q} + \frac{\partial f(x_1, x_2)}{\partial x_2} \frac{\partial x_2}{\partial q} = \frac{df}{dq} = 1$
 Thus we have $\frac{\partial C}{\partial q} = \lambda$

Question 3.1

$$\begin{aligned} \max_{x_1, x_2} x_1 x_2 & & (14) \\ \text{s.t. } p_1 x_1 + p_2 x_2 &= I \end{aligned}$$

First order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_1 + \lambda p_1 = 0 \quad (15)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_2 + \lambda p_2 = 0 \quad (16)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - I = 0 \quad (17)$$

Solve the first order conditions we get

$$x_1 = \frac{I}{2p_1} = \frac{4}{p_1}$$

Question 3.2

$$\Delta CS = \int_{p_1^0}^{p_1^1} x_1 dp_1 = \int_1^2 \frac{4}{p_1} dp_1 = 4 \ln(p_1)_1^2 = 4(\ln 1 - \ln 2)$$

Since it's a price decrease, consumer surplus will be positive thus $\Delta CS = 4 \ln 2$

We need to compare the increase in consumer surplus with the increase in cost. Therefore if $4 \ln 2 > 2$, then it's a good idea. Otherwise it's a bad policy.

Question 4.1

An actuarially fair insurance policy is one in which:

$$\frac{1}{2}P + \frac{1}{2}(P - C) = 0$$

This expression can be solved to find that $P = \frac{1}{2}C$, which can be used to find the budget constraint.

$$\begin{aligned}\frac{1}{2}x_1 + \frac{1}{2}x_2 &= \frac{1}{2}(w_1 - P) + \frac{1}{2}(w_2 - P + C) \\ &= \frac{1}{2}w_1 + \frac{1}{2}w_2 - P + \frac{1}{2}C \\ &= \frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{1}{2}C + \frac{1}{2}C \\ &= \frac{1}{2}w_1 + \frac{1}{2}w_2\end{aligned}$$

Note that if you wanted to use the notation from class, you could write income in state 1 (the good state) as $w_1 = w$ and income in state 2 (the bad state) as $w_2 = w - L$, which also implies that $w_1 > w_2$.

The utility maximization problem is:

$$\max_{x_1, x_2} \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2} \quad \text{subject to} \quad \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}w_1 + \frac{1}{2}w_2$$

The first order conditions are:

$$\begin{aligned}\frac{1}{4\sqrt{x_1}} + \lambda\frac{1}{2} &= 0 \\ \frac{1}{4\sqrt{x_2}} + \lambda\frac{1}{2} &= 0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 &= \frac{1}{2}w_1 + \frac{1}{2}w_2\end{aligned}$$

Dividing the first equation by the second gives that $x_1 = x_2$, and plugging this into the budget constraint gives that:

$$\begin{aligned}x_1 &= \frac{1}{2}w_1 + \frac{1}{2}w_2 \\ x_2 &= \frac{1}{2}w_1 + \frac{1}{2}w_2\end{aligned}$$

Question 4.2

If the insurance company earns an expected payoff of Δ , then:

$$\frac{1}{2}P + \frac{1}{2}(P - C) = \Delta$$

This expression can be solved to find that $P = \frac{1}{2}C + \Delta$, which can be used to find the budget constraint:

$$\begin{aligned}\frac{1}{2}x_1 + \frac{1}{2}x_2 &= \frac{1}{2}w_1 + \frac{1}{2}w_2 - P + \frac{1}{2}C \\ &= \frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{1}{2}C - \Delta + \frac{1}{2}C \\ &= \frac{1}{2}w_1 + \frac{1}{2}w_2 - \Delta\end{aligned}$$

The consumer's utility maximization problem is:

$$\max_{x_1, x_2} \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2} \quad \text{subject to} \quad \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}w_1 + \frac{1}{2}w_2 - \Delta$$

To fully insure would be for $x_1 = x_2$. We would expect the consumer fully insure because the budget constraint has shifted inwards by Δ , but the slope remains the same from problem 4.1. This means the relationship between x_1 and x_2 should be the same as in problem 4.1 ($x_1 = x_2$). It is not true that the consumer fully insures only because they are risk averse. If the insurance company does something to shift the slope of the budget constraint, the consumer would not fully insure (even if they were risk averse).

Question 4.3

The definition of convex preferences is for bundles $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$:

$$x \sim x' \implies \lambda x + (1 - \lambda)x' \succeq x.$$

There are several ways to show that this is the case. The easiest way to show this is to notice that preferences are convex if the utility function $u(x) = \sqrt{x}$

is concave (this is a sufficient, but not necessary condition). You can show this by proving that

$$\sqrt{\lambda x + (1 - \lambda)x'} \geq \lambda\sqrt{x} + (1 - \lambda)\sqrt{x'}$$

which is done by squaring both sides and simplifying. Or you can show that the second derivative is negative:

$$u''(x) = -\frac{1}{4}x^{-\frac{3}{2}} < 0$$

Another method is to show that $U(x_1, x_2, \pi_1, \pi_2)$ is quasiconcave:

$$U(\lambda x + (1 - \lambda)x') \geq \min\{U(x), U(x')\}$$

The final (and most difficult) method is to show directly that the definition of convex preferences holds:

$$\frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2} = \frac{1}{2}\sqrt{x'_1} + \frac{1}{2}\sqrt{x'_2} \implies \frac{1}{2}\sqrt{\lambda x_1 + (1 - \lambda)x'_1} + \frac{1}{2}\sqrt{\lambda x_2 + (1 - \lambda)x'_2} \geq \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2}$$

From the inequality, we get that:

$$\begin{aligned} \frac{1}{2}\sqrt{\lambda x_1 + (1 - \lambda)x'_1} + \frac{1}{2}\sqrt{\lambda x_2 + (1 - \lambda)x'_2} &\geq \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2} \\ &= \lambda\frac{1}{2}\sqrt{x_1} + \lambda\frac{1}{2}\sqrt{x_2} + (1 - \lambda)\frac{1}{2}\sqrt{x_1} + (1 - \lambda)\frac{1}{2}\sqrt{x_2} \\ &= \lambda\frac{1}{2}\sqrt{x_1} + \lambda\frac{1}{2}\sqrt{x_2} + (1 - \lambda)\frac{1}{2}\sqrt{x'_1} + (1 - \lambda)\frac{1}{2}\sqrt{x'_2} \end{aligned}$$

where the last line comes from plugging in $\frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2} = \frac{1}{2}\sqrt{x'_1} + \frac{1}{2}\sqrt{x'_2}$.

Now we want to show that:

$$\sqrt{\lambda x_1 + (1 - \lambda)x'_1} + \sqrt{\lambda x_2 + (1 - \lambda)x'_2} \geq \lambda\sqrt{x_1} + (1 - \lambda)\sqrt{x'_1} + \lambda\sqrt{x_2} + (1 - \lambda)\sqrt{x'_2}$$

This is true if we show that both of the following are true:

$$\begin{aligned} \sqrt{\lambda x_1 + (1 - \lambda)x'_1} &\geq \lambda\sqrt{x_1} + (1 - \lambda)\sqrt{x'_1} \\ \sqrt{\lambda x_2 + (1 - \lambda)x'_2} &\geq \lambda\sqrt{x_2} + (1 - \lambda)\sqrt{x'_2} \end{aligned}$$

Looking at the first inequality:

$$\begin{aligned} \sqrt{\lambda x_1 + (1 - \lambda)x'_1} &\geq \lambda\sqrt{x_1} + (1 - \lambda)\sqrt{x'_1} \\ \lambda x_1 + (1 - \lambda)x'_1 &\geq \lambda^2 x_1 + 2\lambda(1 - \lambda)\sqrt{x_1 x'_1} + (1 - \lambda)^2 x'_1 \\ \lambda(1 - \lambda)x_1 - 2\lambda(1 - \lambda)\sqrt{x_1 x'_1} + \lambda(1 - \lambda)x'_1 &\geq 0 \\ x_1 - 2\sqrt{x_1 x'_1} + x'_1 &\geq 0 \\ (\sqrt{x_1} - \sqrt{x'_1})^2 &\geq 0 \quad \text{TRUE} \end{aligned}$$

where the second line comes from squaring both sides. The same thing can be done to show the second inequality holds as well.