

Microeconomics

Question 1.1

We want to solve the following utility maximization problem

$$\begin{aligned} & \max_{x_1, x_2} x_1^2 x_2 \\ \text{s.t. } & x_1 + \frac{x_2}{1+r} = I_1 + \frac{I_2}{1+r} \end{aligned}$$

The Lagrange for above maximization problem is

$$\mathcal{L} = x_1^2 x_2 + \lambda \left(x_1 + \frac{x_2}{1+r} - I_1 - \frac{I_2}{1+r} \right)$$

First order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 x_2 + \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1^2 + \lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_1 + \frac{x_2}{1+r} - I_1 - \frac{I_2}{1+r} = 0 \quad (3)$$

Dividing equation (1) by equation (2) to eliminate the Lagrange multiplier, we get

$$\frac{2x_2}{x_1} = 1 + r$$

which yields

$$x_2 = \frac{(1+r)x_1}{2} \quad (4)$$

Substitute (4) into equation (3) we get

$$x_1 + \frac{(1+r)x_1}{2(1+r)} - I_1 - \frac{I_2}{1+r} = 0 \quad (5)$$

Together with (4) and (5) we get

$$x_1 = \frac{2I_1}{3} + \frac{2I_2}{3(1+r)} \quad (6)$$

$$x_2 = \frac{I_1(1+r)}{3} + \frac{I_2}{3} \quad (7)$$

Question 1.2

When $I_1 = I_2 = 100$ and $r = 0.1$, we have

$$x_1 = \frac{2}{3} \times 100 + \frac{2}{3} \times \frac{100}{1.1} > I_1$$

Therefore this person is a borrower

Question 1.3

From equation (6) and (7), we see that when r increases, x_1 decreases and x_2 increases. Therefore increase in interest rate will decrease current consumption and increase future consumption. Since $x_1 - I_1$ decreases when x_1 decreases, the amount this person borrows will decrease as r increases.

Question 2.1

$$L = x_1x_2^3 + \lambda(p_1x_1 + p_2x_2 - I)$$

First order conditions:

$$\begin{aligned}\frac{dL}{dx_1} &= x_2^3 + \lambda p_1 = 0 \\ \frac{dL}{dx_2} &= 3x_1x_2^2 + \lambda p_2 = 0 \\ \frac{dL}{d\lambda} &= p_1x_1 + p_2x_2 - I = 0\end{aligned}$$

Question 2.2

Bordered Hessian:

$$\begin{bmatrix} L_{11} & L_{12} & L_{1\lambda} \\ L_{21} & L_{22} & L_{2\lambda} \\ L_{\lambda 1} & L_{\lambda 2} & L_{\lambda\lambda} \end{bmatrix} = \begin{bmatrix} 0 & 3x_2^2 & p_1 \\ 3x_2^2 & 6x_1x_2 & p_2 \\ p_1 & p_2 & 0 \end{bmatrix}$$

The second order conditions are:

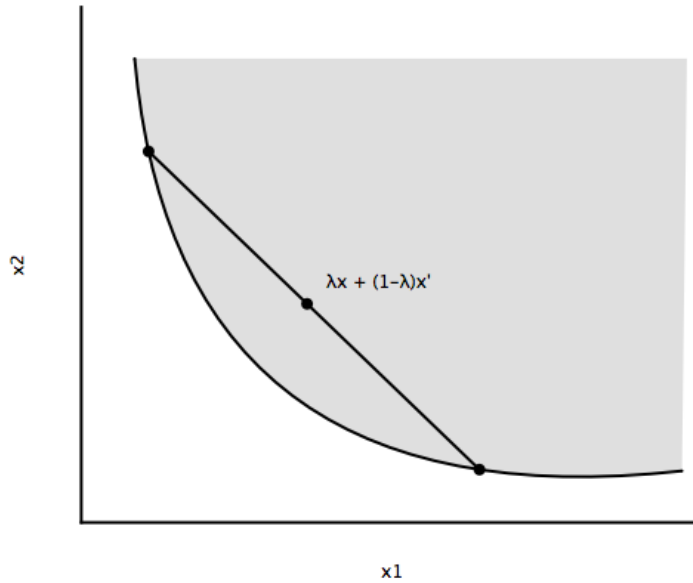
$$\begin{aligned}\begin{vmatrix} 0 & p_1 \\ p_1 & 0 \end{vmatrix} &< 0 \\ \begin{vmatrix} 0 & 3x_2^2 & p_1 \\ 3x_2^2 & 6x_1x_2 & p_2 \\ p_1 & p_2 & 0 \end{vmatrix} &> 0.\end{aligned}$$

Question 2.3

A function U is quasiconcave if, for all x, x' and $\lambda \in (0, 1)$, we have

$$U(\lambda x + (1 - \lambda)x') \geq \min[U(x), U(x')], \quad (8)$$

An alternate definition is that U is quasiconcave if and only if it has convex upper contour sets. The graph below of the indifference curve for $U(x_1, x_2) = x_1x_2^3$ shows that it is quasiconcave using the second definition (the gray area is the upper contour set and the labeled point shows that the upper contour set is convex):



Question 2.4

Setting the MRS equal to the price ratio gives:

$$\frac{x_2^3}{3x_1x_2^2} = \frac{p_1}{p_2}$$

$$x_2 = \frac{3p_1x_1}{p_2}$$

Plug this into the budget constraint to get:

$$p_1x_1 + p_2 \left(\frac{3p_1x_1}{p_2} \right) = I$$

$$4x_1p_1 = I$$

$$x_1 = \frac{I}{4p_1}$$

Plug this into the expression for x_2 (the question doesn't ask you to find this, but it's useful to have this to find the indirect utility function later):

$$x_2 = \frac{3p_1x_1}{p_2}$$

$$\begin{aligned}
 &= \frac{3p_1 I}{p_2^4 p_1} \\
 &= \frac{3I}{4p_2}
 \end{aligned}$$

Question 2.5

The price elasticity of demand for good 1 is $-\frac{p_1}{x_1} \frac{dx_1}{dp_1}$. First, find $\frac{dx_1}{dp_1}$:

$$\frac{dx_1}{dp_1} = -\frac{I}{4p_1^2}$$

Plug this into the expression for the price elasticity, as well as plugging in the demand equation for good 1 from question 2.4:

$$e = -\frac{4p_1^2}{I} \left(-\frac{I}{4p_1^2} \right) = 1$$

This means good 1 has unit elasticity; the percent change in the quantity of good 1 demanded is exactly the same as the percent change in price.

Question 2.6

To find the indirect utility function, plug the demand functions into the original utility function:

$$V(p_1, p_2, I) = \frac{I}{4p_1} \left(\frac{3I}{4p_2} \right)^3 = \frac{3^3 I^4}{4^4 p_1 p_2^3}$$

Question 2.7

The expenditure minimization problem is:

$$\min p_1 x_1 + p_2 x_2 - I \quad \text{subject to} \quad x_1 x_2^3 = \bar{U}$$

$$L = p_1 x_1 + p_2 x_2 - I + \lambda(x_1 x_2^3 - \bar{U})$$

The first order conditions are exactly the same as those found in the utility maximization problem:

$$x_2 = \frac{3p_1 x_1}{p_2}$$

Plug this into the utility constraint:

$$\begin{aligned} x_1 \left(\frac{3p_1 x_1}{p_2} \right)^3 &= \bar{U} \\ x_1^4 &= \frac{p_2^3 \bar{U}}{3^3 p_1^3} \\ x_1^c &= \frac{p_2^{\frac{3}{4}} \bar{U}^{\frac{1}{4}}}{3^{\frac{3}{4}} p_1^{\frac{3}{4}}} \end{aligned}$$

The Slutsky equation is $\frac{dx_1}{dp_1} = \frac{dx_1^c}{dp_1} - x_1 \frac{dx_1}{dI}$. To verify this holds, solve for each component of the Slutsky equation. The left hand side is equal to:

$$\frac{dx_1}{dp_1} = -\frac{I}{4p_1^2}$$

The first term of the right hand side is:

$$\frac{dx_1^c}{dp_1} = -\frac{3p_2^{\frac{3}{4}} \bar{U}^{\frac{1}{4}}}{4(3^{\frac{3}{4}})p_1^{\frac{7}{4}}}$$

To get rid of \bar{U} , notice that \bar{U} is equal to the indirect utility function, $V(p_1, p_2, I)$. This means:

$$\bar{U}^{\frac{1}{4}} = [V(p_1, p_2, I)]^{\frac{1}{4}} = \left[\frac{3^3 I^4}{4^4 p_1 p_2^3} \right]^{\frac{1}{4}} = \frac{3^{\frac{3}{4}} I}{4p_1^{\frac{1}{4}} p_2^{\frac{3}{4}}}$$

Now plug this back into the expression for $\frac{dx_1^c}{dp_1}$:

$$\begin{aligned} \frac{dx_1^c}{dp_1} &= -\frac{3p_2^{\frac{3}{4}}}{4(3^{\frac{3}{4}})p_1^{\frac{7}{4}}} \left(\frac{3^{\frac{3}{4}} I}{4p_1^{\frac{1}{4}} p_2^{\frac{3}{4}}} \right) \\ &= -\frac{3I}{16p_1^2} \end{aligned}$$

The second term of the right hand side equals:

$$x_1 \frac{dx_1}{dI} = \frac{I}{4p_1} \left(\frac{1}{4p_1} \right) = \frac{I}{16p_1^2}$$

Putting together the two terms shows that the right hand side equals:

$$\frac{dx_1^c}{dp_1} - x_1 \frac{dx_1}{dI} = -\frac{3I}{16p_1^2} - \frac{I}{16p_1^2} = -\frac{I}{4p_1^2}$$

which is the same as the left hand side.

Question 3.1

The problem is one that maximize utility subject to a budget constraint:

$$\begin{aligned} & \max_{c,h} U(c, h) \\ \text{s.t. } & c + wh = wT \end{aligned}$$

The first order conditions are:

$$\begin{aligned} \frac{dL}{dc} : & \quad \frac{dU}{dc} = \lambda \\ \frac{dL}{dh} : & \quad \frac{dU}{dh} = \lambda w \\ \frac{dL}{d\lambda} : & \quad c + wh - wT = 0 \end{aligned}$$

It simplifies to:

$$\frac{MU_c}{MU_h} = \frac{1}{w}$$

Question 3.2

The new budget constraint becomes:

$$\begin{aligned} & \max_{c,h} U(c, h) \\ \text{st } & c + (1 - t)wh = (1 - t)wT \end{aligned}$$

The new first order condition becomes:

$$\begin{aligned} \frac{dL}{dc} : & \quad \frac{dU}{dc} = \lambda \\ \frac{dL}{dh} : & \quad \frac{dU}{dh} = \lambda w(1 - t) \\ \frac{dL}{d\lambda} : & \quad c + (1 - t)wh - (1 - t)wT = 0 \end{aligned}$$

It simplifies to:

$$\frac{MU_c}{MU_h} = \frac{1}{(1-t)w}$$

Question 3.3

The new budget constraint becomes $c + wh = wT - L$, and the first order condition is the same as in part 3.1

Question 3.4

When the two tax revenues are the same, the condition is $L = tw(T - h^*)$, where h^* is the optimal amount of leisure chosen by the consumer under the income tax.

The budget line under the income tax has the same h-intercept as the original one but it has a steeper slope. The consumer will choose a bundle within the feasible set (the area between the budget constraint and the axis) that maximizes her utility. Call that bundle $x^* = (c^*, h^*)$.

The budget line under the lumpsum tax is a line that is a parallel shift of the original budget set towards the origin. Under the assumption that $L = tw(T - h)$, the consumer should have the same post-tax income under both taxation system, and the bundle x^* should be feasible in both cases. Therefore, the budget line under the lumpsum tax should pass through the bundle x^* . See the attached graph for an example.

We argue that the consumer is able to achieve at least as much utility under the lumpsum tax as they can under the income tax. The logic is that given the consumer chooses x^* under the income tax, x^* must be as good as any other bundle in its feasible set. The same consumer, when subjected to the lumpsum tax, will not choose any of these "other bundles" either, because x^* is feasible under the lumpsum tax, and x^* is better than other bundles in the feasible set of the income tax. It means that the consumer, when subjected to the lumpsum tax, will either choose x^* , or another bundle that is feasible under the lumpsum tax but outside of the feasible set of the income tax. If she picks x^* , then the consumer receives the same utility under the two tax systems. On the other hand, if she picks an outside bundle, she

must have a higher utility (otherwise she would have picked x^* instead). In both cases, we know that the consumer under the lumpsum is at least as good as, or better than the one under the income tax.

