Question 1

1.1 The Lagrange function is:

$$\mathcal{L} = x_1 - \frac{1}{2}x_1^2 + x_2 + \lambda(I - p_1x_1 - p_2x_2).$$

First-order conditions:

$$\begin{aligned} \mathcal{L}_1 &= 1 - x_1 - \lambda p_1 &&= 0 \\ \mathcal{L}_2 &= 1 - \lambda p_2 &&= 0 \\ \mathcal{L}_\lambda &= I - p_1x_1 - p_2x_2 &&= 0. \end{aligned}$$

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1.2 The Bordered Hessian is:

$$\mathbf{H} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{1\lambda} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{2\lambda} \\ \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda 2} & \mathcal{L}_{\lambda\lambda} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix}.$$

Let

$$\mathbf{K} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{1\lambda} \\ \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda\lambda} \end{pmatrix} = \begin{pmatrix} -1 & -p_1 \\ -p_1 & 0 \end{pmatrix}.$$

The second-order conditions are:

$$\begin{aligned} -det\,(\mathbf{K}) &> 0 \\ det\,(\mathbf{H}) &> 0. \end{aligned}$$

1.3 We need to solve the first-order conditions for the demand function of good 1 in order to solve this problem. Doing this gives you

$$x_1 = 1 - \frac{p_1}{p_2}.$$

(Notice that demand for x_1 is only positive if $p_2 > p_1$. Otherwise, we have a corner solution.)

The own price elasticity of demand for good 1 is

$$e = -\frac{p_1}{x_1}\frac{dx_1}{dp_1} = \frac{p_1}{p_2 - p_1}.$$

Demand is elastic if $e > 1$, or

$$\frac{p_1}{p_2 - p_1} > 1.$$

With some algebra, this inequality reduces to

$$p_1 > \frac{p_2}{2}.$$

Demand is inelastic if $e < 1$. A similar argument shows that this is the case if

$$p_1 < \frac{p_2}{2}.$$

Question 2

Question 2.1

With a subsidy of S the individual faces the problem of maximizing $U(x_1, x_2) = (x_1 - 10)x_2$ subject to the constraint that $x_1 + x_2 = I + S$. The associated Lagrangian is then

$$L(x_1, x_2, \lambda) = (x_1 - 10)x_2 + \lambda(x_1 + x_2 - I - S).$$

Taking first order conditions we get

$$\frac{dL}{dx_1} = x_2 + \lambda = 0,$$

$$\frac{dL}{dx_2} = x_1 - 10 + \lambda = 0,$$

$$\frac{dL}{d\lambda} = x_1 - x_2 - I - S = 0.$$

Combining the first two equations gives $x_1 = x_2 + 10$. Plugging this into the third condition gives $2x_2 - 10 - I - S = 0$, and so

$$x_2 = \frac{I + S}{2} - 5.$$

Therefore,

$$\frac{dx_2}{dS} = \frac{1}{2} > 0,$$

so the consumption of good 2 is increasing in the subsidy S.

Question 2.2

See Figure 1. Under the original budget constraint, the individual chooses among bundles (x_1, x_2) with $x_1 + x_2 = I$. With the gross subsidy, S, the budget constraint becomes $x_1 + x_2 = I + S$. With the proportional subsidy introduced in this question the budget constraint is $(1 - t)x_1 + x_2 = I$. For any subsidy, t, we can calculate the bundle the individual would purchase. Call this x^t. This bundle gives the individual utility $U^t = U(x^t)$. Note that the cost to the government is then tx_1^t. In order for the two programs to cost the government the same amount we must have $tx_1^t = S$. Hence we must have that $x_1^t + x_2^t = I + tx_1^t = I + S$, so the optimal bundle for the individual must be the point where the budget lines associated with the two types of subsidies intersect.

In addition, since x^t is the optimal bundle for the individual under the proportional subsidy, the indifference curve associated with utility level u^t must be tangent to the budget line with the proportional subsidy at this point. Because the budget line associated with the fixed subsidy, S, is steeper this indifference curve crosses the budget line with the fixed subsidy at two points, x^t and $x^{'}$. Any point on the line between x^t and $x^{'}$ would then give the individual utility higher than u^t. Therefore, the individual's utility is higher with the direct subsidy S than with the proportional subsidy t.

Question 3

Question 3.1

The budget constraint is $c = w(T - h - kc)$.

Question 3.2

The maximization problem is:

$$\max_{c,h} \quad ch \qquad \text{subject to} \qquad c = w(T - h - kc)$$

The lagrangian from this maximization problem is:

$$\mathcal{L} = ch + \lambda(c - w(T - h - kc))$$

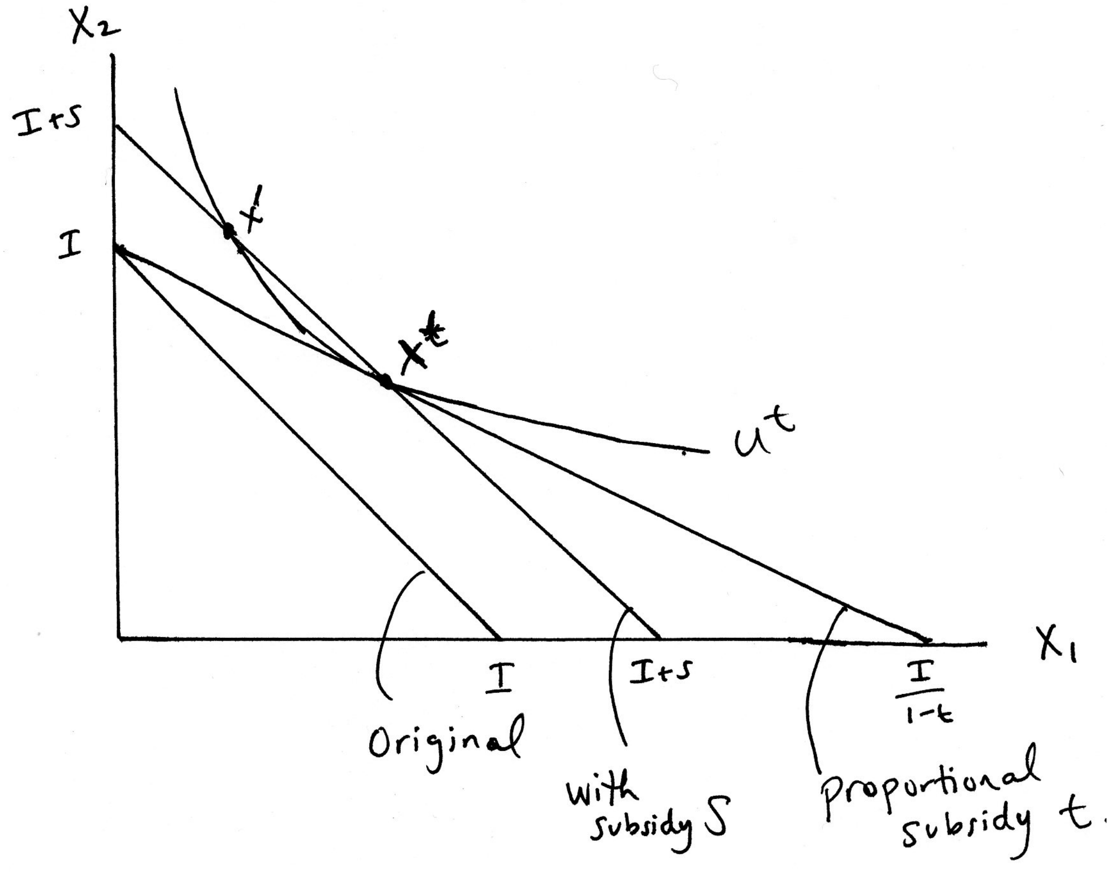
The first order conditions are then:

$$c + \lambda w = 0$$
$$h + \lambda + \lambda wk = 0$$

Solving these two equations for c gives the following demand function for consumption:

$$c = \frac{wT}{2(1 + wk)}$$

Figure 1



X_2
I+S
I
x'
x*
u^t
X_1
I
I+S
I/(1-t)
Original
with subsidy S
proportional subsidy t.

Question 3.2

$$\frac{dc}{dk} = -\frac{2w^2T}{4(1+wk)^2} < 0$$

Since $\frac{dc}{dk} < 0$, the optimal consumption of c increases if k decreases. Another way of thinking about this is that since k is in the denominator, when k gets smaller, the denominator gets smaller, and the entire expression for c gets bigger. The common saying "time is money" does make sense.

Question 4

- 1. The maximization problem is:

$$\max_{x_1,x_2} x_1^2 x_2 \text{ subject to } \quad x_1 + \frac{x_2}{1+r} = I_1 \tag{1}$$

The first order conditions are:

$$2x_1x_2 - \lambda = 0 \tag{2}$$

$$x_1^2 - \frac{\lambda}{1+r} = 0 \tag{3}$$

$$I_1 - x_1 - \frac{x_2}{1+r} = 0 \tag{4}$$

Solving the first two first order conditions gives $x_2 = \frac{(1+r)}{2}x_1$. Plug this into the budget constraint. After more algebra you come out with the demand function:
$x_1(1, \frac{1}{1+r}, I_1) = \frac{2}{3}I_1$
A similar calculation gives the demand for x_2[1]:
$x_2(1, \frac{1}{1+r}, I_1) = \frac{(1+r)}{3}I_1$

2. The maximization problem is:

$$\min_{x_1,x_2} x_1 + \frac{x_2}{1+r} \qquad \text{subject to} \qquad x_1^2 x_2 = \overline{U} \tag{5}$$

The first order conditions are:

$$-1 - 2\lambda x_1 x_2 = 0 \tag{6}$$

$$-\frac{1}{(1+r)} - \lambda x_1^2 = 0 \tag{7}$$

[1] finding x_2 is not required

$$\overline{U} - x_1^2 x_2 = 0 \tag{8}$$

Manipulating the first two equations then dividing them we get $x_2 = \frac{2}{1+r}x_1$.Put this equation into the third first-order condition. Algebra gives us compensated demand for x_1, which is:

$x_1^c(1, \frac{1}{1+r}, \overline{U}) = (\frac{2\overline{U}}{1+r})^{\frac{1}{3}}$ and,

$x_2^c(1, \frac{1}{1+r}, \overline{U}) = (\frac{2}{1+r})^{\frac{2}{3}} \overline{U}^{\frac{1}{3}}$ [2]

3. The first task here is to make sense of this question. The idea is to think about the Slutzky equation and to see what one might do in this case. There are a number of possibilities, any one of which sufficed to obtain at least most of the credit on this problem. One common response (but by no means the only good one) was to introduce a price p_1 for good x_1 and re-solve parts 1 and 2. After finding the Marshallian and compensated demand functions for x_1, take the derivatives. Then substitute $p_1 = 1$.

   The Sluksky Equation is:

$$\frac{dx_1}{dp_1} = \frac{dx_1^c}{dp_1} - \frac{dx_1}{dI}x_1.$$

   Solving the utility maximization problem gives: $x_1(p_1, \frac{1}{1+r}, I_1) = \frac{2I_1}{3p_1}$

   Solving the expenditure minimization problem gives: $x_1^c(p_1, \frac{1}{1+r}, \overline{U}) = (\frac{2\overline{U}}{(1+r)p_1})^{\frac{1}{3}}$

   $\frac{dx_1}{dp_1} = -\frac{2I_1}{3p_1^2}, \frac{dx_1}{dI} = \frac{2}{3p_1}$ and $\frac{dx_1^c}{dp_1} = -\frac{1}{3}(\frac{2\overline{U}}{(1+r)p_1^4})^{\frac{1}{3}}$

   Now set $p_1 = 1$ and you get:

   $\frac{dx_1}{dp_1} = -\frac{2I_1}{3}, \frac{dx_1}{dI} = \frac{2}{3}$ and $\frac{dx_1^c}{dp_1} = -\frac{1}{3}(\frac{2\overline{U}}{(1+r)})^{\frac{1}{3}}$

   Plugging these into the Slutsky Equation gives:

   $-\frac{1}{3}(\frac{2\overline{U}}{(1+r)})^{\frac{1}{3}} = -\frac{2I_1}{3} - \frac{2}{3} . \frac{2}{3} I_1$

   You can (but would not need to on an exam) show that this equality holds with enough algebra. If you are not convinced, pick some sample numbers and check that it is true.

[2] finding x_2 is not required