Macroeconomics

Partial Answer Key

Section 1. (Suggested Time: 45 Minutes) *For 3 of the following 6 statements*, state whether the statement is **true**, **false**, or **uncertain**, and give a complete and convincing explanation of your answer. **Note:** Such explanations typically appeal to specific macroeconomic models.

1. Most of the expenditures in President Obama's economic stimulus plan will be made some time in the future. As a result, the plan will have almost no effect on the economy today.

2. Sticky-price Neo-Keynesian models show that the IS-LM model is fully consistent with rational economic behavior.

3. Putting money in the utility function is never a valid modeling technique in an infinite horizon environment.

4. Search is irrelevant for modelling retail markets: "when I want shoes I go to a shoe store; when I want coffee I go to a coffee shop".

5. In either the Keynesian macroeconomic model or the real business cycle model, banks play no role. Each consumer maximizes utility subject to his budget constraint, and each firm maximizes profit. Banks and other financial intermediaries channel funds from savers to investors, but this activity is invisible in each model.

6. Aggregate consumption demand responds to stock price fluctuations in accord with economic theory. A drop in the stock market reduces the wealth of investors, and each investor reduces his consumption in line with how the fall in wealth tightens his budget constraint. Consequently the ups and downs of the stock market are the key to understanding the business cycle

Section 2. (Suggested Time: 2 Hours, 15 minutes) Answer any 3 of the following 4 questions.

7. We are considering a stochastic growth model where technology is observed with a lag. The representative price-taking firm solves

$$\max_{L_t \geq 0} E\left(\Pi_t | I_t\right) = E\left(Y_t | I_t\right) - W_t L_t,$$

$$Y_t = A_t L_t^{\eta}, \quad 0 < \eta < 1. \tag{PRF}$$

The representative household solves the following problem:

$$\max_{\{C_t, B_{t+1}, L_t\}_{t=0}^{\infty}} E\left(\sum_{t=0}^{\infty} \beta^t \left[C_t - \chi \frac{1}{1+\gamma} L_t^{1+\gamma}\right] \Bigg| I_0\right),$$

$$\gamma \geq 0, \quad \chi > 0, \quad 0 < \beta < 1,$$

$$s.t. \quad C_t + B_{t+1} = (1 + r_t) B_t + W_t L_t + \Pi_t, \tag{FBC}$$

and the usual boundary conditions. Note that there is no storage technology; the risk-free bonds (B_t) represent trades between consumers.

The log of productivity follows an AR(1) process

$$a_t \equiv \ln(A_t) = \phi a_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1, \tag{TS}$$

where $\{\varepsilon_t\}$ is an i.i.d. shock. However, at time t households and firms only observe $Z_t = A_t U_t$, where $u_t \equiv \ln(U_t)$ is i.i.d. measurement error. Assume further that

$$a_t \sim \mathcal{N}(0, \sigma_a^2); \quad u_t \sim \mathcal{N}(0, \sigma_u^2); \quad cov(a_t, u_s) = 0, \quad \forall t, s.$$

Agents do observe lagged values of A, however, so that the time-t information set is

$$I_t = \{W_t, r_t, B_t, Z_t, A_{t-1}\},$$

which implies that the firm predicts A_t (and picks L_t) on the basis of Z_t and A_{t-1}. Note that the household must pick C_t before profits, Π_t, are realized..

(a) The firm's problem can be written as

$$\max_{L_t \geq 0} E\left(A_t | I_t\right) L_t^{\eta} - W_t L_t,$$

and the first order condition for profit maximization is

$$\eta E\left(A_t | I_t\right) L_t^{\eta - 1} = W_t. \tag{PM}$$

The Bellman equation for the household can be written as

$$V\left(B_t, Z_t, A_{t-1}\right) = \max_{C_t \geq 0, L_t \in [0,1]} C_t - \chi \frac{1}{1+\gamma} L_t^{1+\gamma}$$
$$+ \beta E\left(V\left((1 + r_t) B_t + W_t L_t + \Pi_t - C_t, Z_{t+1}, A_t\right) | I_t\right).$$

The FOC are

$$\begin{aligned} 1 &= \beta E\left(\left.\frac{\partial V[t+1]}{\partial B_{t+1}}\right| I_t\right), \\ \chi L_t^{\gamma} &= \beta E_t\left(\left.\frac{\partial V[t+1]}{\partial B_{t+1}}\right| I_t\right) W_t. \end{aligned}$$

Because (following Benveniste-Scheinkman)

$$\frac{\partial V[t]}{\partial B_t} = \beta E_t\left(\left.\frac{\partial V[t+1]}{\partial B_{t+1}}\right| I_t\right)(1+r_t) = (1+r_t),$$

these FOC reduce to

$$\begin{aligned} W_t &= \chi L_t^{\gamma}, \\ 1 &= \beta(1+r_{t+1}). \end{aligned} \tag{EE}$$

Using equation (PM) to eliminate wages, the labor allocation condition becomes

$$L_t = \left(\frac{\eta}{\chi}\right)^{\theta} E\left(A_t|\, I_t\right)^{\theta}, \qquad \theta = \frac{1}{\gamma+1-\eta} > 0. \tag{LL}$$

The resource constraint can be written as

$$C_t = Y_t = A_t L_t^{\eta}, \tag{RC}$$

with Y_t following equation (PRF). This equation can either be derived directly as a resource constraint, or found by combining the household's budget constraint with the definition of profits, and imposing the equilibrium allocation $B_t = 0$.

(b) Define $\widetilde{a}_t = E(a_t|\, I_t)$. Because ε_t and u_t are independent, normally-distributed variables, the vector (a_t, z_t, a_{t-1}) follows a multivariate normal distribution and we can use standard linear projection formula. It can be shown that $\widetilde{a}_t$ is

$$\widetilde{a}_t = \lambda(\varepsilon_t + u_t) + \phi a_{t-1}, \quad \lambda = \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2 + \sigma_u^2}, \tag{PRJ}$$

(see the Spring 2008 exam) and that v_t, the forecast error for a_t, is

$$v_t \equiv a_t - \widetilde{a}_t = (1-\lambda)\,\varepsilon_t - \lambda u_t.$$

Let lower-case letters with carats "$\widehat{\ }$" denote deviations of logged variables around their steady state values. It follows from equation (LL) that

$$\begin{aligned} \exp\left(\widehat{\ell}_t\right) &\equiv \frac{L_t}{L_{ss}} = \left(\frac{\eta}{\chi}\right)^{\theta} E\left(A_t|\, I_t\right)^{\theta} \Big/ \left[\left(\frac{\eta}{\chi}\right)^{\theta} \cdot 1\right] \\ &= E\left(\exp(a_t)|\, I_t\right)^{\theta}. \end{aligned}$$

Logging both sides yields

$$\widehat{\ell}_t = \theta \ln\left(E\left(\exp(a_t)|\, I_t\right)\right),$$

which, upon imposing the approximation $\ln\left(E\left(A_t|\, I_t\right)\right) \approx E\left(\ln(A_t)|\, I_t\right)$, simplifies to

$$\widehat{\ell}_t \approx \theta E\left(a_t|\, I_t\right) = \theta\widetilde{a}_t. \tag{LM}$$

Proceeding similarly with equation (PRF), we have

$$\begin{aligned}
\widehat{y}_t &= a_t + \eta\widehat{\ell}_t \\
&\approx a_t + \eta\theta\widetilde{a}_t \\
&\equiv a_t + \mu\widetilde{a}_t = a_t + \mu(a_t - v_t) = (1+\mu)\, a_t - \mu v_t \\
&= (1+\mu)\, a_t - \mu\left[(1-\lambda)\,\varepsilon_t - \lambda u_t\right]. \qquad \text{(PRF}'\text{)}
\end{aligned}$$

(c) Let's consider the economy's response to a technology shock.

1. Suppose that $\phi = 0.9$, $\gamma = 1/3$, $\eta = 2/3$ and $\lambda = 1/2$. Using equations (LL) and (PRF′), it follows that

$$\begin{aligned}
\theta &= \frac{1}{\gamma + 1 - \eta} = \frac{3}{2}, \\
\mu &= \eta\theta = 1, \\
\widehat{y}_t &= 2a_t - \frac{1}{2}\varepsilon_t + \frac{1}{2}u_t. \qquad \text{(PRF}''\text{)}
\end{aligned}$$

2. Suppose further that: $\widehat{y}_0 = a_0 = \varepsilon_0 = 0$; $u_0 = u_1 = u_2 ..... = 0$; $\varepsilon_1 = 1$; $\varepsilon_2 = \varepsilon_3 = ... = 0$. Using equations (TS) and (PRF″) produces the following impulse response functions:

t	ε_t	a_t	u_t	$\widehat{y}_t$
0	0.00	0.00	0.00	0.00
1	1.00	1.00	0.00	1.5
2	0.00	0.90	0.00	1.8
3	0.00	0.81	0.00	1.62
4	0.00	0.73	0.00	1.46

3. The hump shape of the output response matches the hump shape exhibited in the data. The hump arises because productivity is not perfectly observed. When the productivity shock first hits, agents attribute part of the shock to measurement error (u_t) and do not fully adjust labor. In the following period, firms observe the improved productivity and increase employment further still. Although productivity is now slightly lower than in the first period, the net effect is that output increases again. In the remaining periods, productivity is perfectly observed, and output monotonically returns to its steady state value.

8. **Overlapping generations with acquired human capital.**

Consider the following economy:

Time: discrete, infinite horizon

Demography: A mass $N_t \equiv N_0(1+n)^t$ of newborns enter in every period. Everyone lives for 2 periods except for the first generation of old people.

Preferences: for the generations born in and after period 0;

$$U_t(c_{1,t}, c_{2,t+1}) = u(c_{1,t}) + \beta u(c_{2,t+1}),$$

where $c_{i,t}$ is consumption in period t and stage i of life, $u(.)$ is increasing strictly concave and twice differentiable $\lim_{c\to 0} u'(c) = \infty$, $\lim_{c\to\infty} u'(c) = 0$. For the initial old generation $\tilde{U}(c_{2,0}) = u(c_{2,0})$.

Productive technology: $f(h, l)$ is a neoclassical production technology which uses human capital and labor to produce a consumption good, f is homogeneous of degree 1, twice differentiable, increasing in both arguments, strictly concave with $\lim_{h\to 0} f_1(h, l) = \infty$ for all $l > 0$ and $\lim_{l\to 0} f_2(h, l) = \infty$ for all $h > 0$.

Human Capital Technology: $h_{t+1} = 1 - l_t$. Time not spent working in youth contributes to human capital available in old age.

Endowments: Everyone has one unit of labor services when young. They can use any fraction of this to work in the first period of life Any unused labor services become human capital for use in the second period of life. (When old, only human capital can be used to earn money for consumption.) The initial old have h_0 units of human capital.

Institutions: There are competitive markets, for labor and human capital. (You can think of a single collectively owned firm which takes wages and interest rates as given.) Using the consumption good as the numeraire, let the per unit wage for each market in period t be w_t^l and w_t^h for labor and human capital respectively.

(a) Write out and solve the problems faced by generation t workers and period t firms in this economy.

Workers born in period t solve

$$\begin{aligned} \max_{l_t, h_{t+1}} \ & u(c_{1,t}) + \beta u(c_{2,t+1}) \\ \text{s.t.} \quad & c_{1,t} \le w_t^l l_t, \\ & c_{2,t+1} \le w_{t+1}^h h_{t+1}, \\ & h_{t+1} = 1 - l_t, \end{aligned}$$

Taking advantage of monotonicity of preferences, and substituting for $c_{1,t}$, $c_{2,t+1}$ and l_t, this becomes

$$\max_{h_{t+1}} u(w_t^l(1 - h_{t+1})) + \beta u(w_{t+1}^h h_{t+1})$$

The maximand is strictly concave. The FOC is:

$$-w_t^l u'(w_t^l(1-h_{t+1})) + \beta w_{t+1}^h u'(w_{t+1}^h h_{t+1}) = 0,$$

which defines the implicit function

$$h_{t+1} = H(w_t^l, w_{t+1}^h).$$

Workers born old in the first period have no decisions to make.

Firms in period t solve:

$$\max_{\tilde{l}_t, \tilde{h}_t} f\left(\tilde{l}_t, \tilde{h}_t\right) - w_t^l \tilde{l}_t - w_t^h \tilde{h}_t.$$

The maximand is strictly concave. The FOC's are:

$$\begin{aligned} \tilde{l}_t &: \quad f_1\left(\tilde{l}_t, \tilde{h}_t\right) = w_t^l, \\ \tilde{h}_t &: \quad f_2\left(\tilde{l}_t, \tilde{h}_t\right) = w_t^h. \end{aligned}$$

(b) Define a competitive equilibrium and solve for the implied law of motion for the per-old person stock of human capital, h_t, in the economy.

Definition: A competitive equilibrium is an allocation, $\{c_{1,t}, c_{2,t}, \}$, a sequence of human labor services and capital investments, $\{l_t, h_t\}$ and prices $\{w_t^l, w_t^h\}$ such that given prices, the allocation solves the workers' and firms' problems and markets clear.

Market clearing (with a single firm) requires

$$\begin{aligned} \tilde{l}_t &= N_t l_t \\ \tilde{h}_t &= N_{t-1} h_t \end{aligned}$$

So after dividing through by N_{t-1}, we have

$$y_t = f([1+n][1-h_{t+1}], h_t),$$

where y_t is per-old-person output, and

$$\begin{aligned} w_t^l &= f_1([1+n][1-h_{t+1}], h_t) \\ w_t^h &= f_2([1+n][1-h_{t+1}], h_t). \end{aligned}$$

This allows us to rewrite the FOC for human capital accumulation as

$$h_{t+1} = H(f_1([1+n][1-h_{t+1}], h_t), f_2([1+n][1-h_{t+2}], h_{t+1}))$$

or

$$\begin{aligned} &\beta f_2([1+n][1-h_{t+2}], h_{t+1}) u'(f_2([1+n][1-h_{t+2}], h_{t+1}) \cdot h_{t+1}) \\ &\quad - f_1([1+n][1-h_{t+1}], h_t) u'(f_1([1+n][1-h_{t+1}], h_t) \cdot [1-h_{t+1}])) = 0. \end{aligned}$$

(c) If $u(c) = \ln c$ and $f(h, l) = Ah^{\alpha} l^{1-\alpha}$, what is the law of motion for h_t?

With log utility,the FOC for human capital accumulation is

$$-w_t^l \frac{1}{w_t^l(1 - h_{t+1})} + \beta w_{t+1}^h \frac{1}{w_{t+1}^h h_{t+1}} = \frac{\beta}{h_{t+1}} - \frac{1}{1 - h_{t+1}} = 0.$$

(d) Maintaining these functional forms, solve for the steady state level of human capital, $\bar{h}$. What can you say about its dynamic properties (i.e. stability, oscillatory)?

The answer to part (c) implies that

$$h_t = \bar{h} = \frac{\beta}{1 + \beta},$$

for all t. So h_t and l_t are constant. There are no dynamical equilibria.

9. **Asset pricing with money.** Consider an economy with a representative consumer who maximizes

$$E_0\left(\sum_{t=0}^{\infty}\beta^t\left[\ln\left(c_t\right)+\alpha\ln(m_t)\right]\right),\quad 0<\beta<1,\quad \alpha>0,$$

where c_t denotes consumption, m_t denotes real balances. and $E_t(\cdot)$ denotes expectations conditional on time-t information.

The sole source of the single non-storable good is an everlasting tree that produces d_t units of the consumption good in period t. At the beginning of time 0, each consumer owns one such tree. Dividends are exogenous and follow a time-invariant Markov process.

The consumer's wealth evolves according to

$$M_t+P_t(c_t+R_t^{-1}b_{t+1}+q_t s_{t+1})+I_t^{-1}N_t=P_t(\tau_t+[q_t+d_t]s_t+b_t)+N_{t-1}+M_{t-1},\quad \text{(NFBC)}$$

where:

M_t is the nominal money supply and P_t is the aggregate price level;

q_t is the real price at time t of a title to all future dividends from the tree, and s_t is the number of trees;

R_t^{-1} is the real price of a risk-free discount bond that pays one unit of consumption at time $t+1$, and b_t is the (real) quantity of such bonds;

I_t^{-1} is the nominal price of a discount bond that pays one unit of money at time $t+1$, and N_t is the (nominal) quantity of such bonds. Note that nominal bonds are dated in the same way as money, rather than stocks or real bonds.

The nominal money supply M_t evolves according to

$$M_t=(1+\theta_t)M_{t-1},\qquad \text{(MS)}$$

with θ_t following an exogenous Markov process. In equilibrium, lump-sum taxes equal seigniorage:

$$\tau_t=\frac{M_t-M_{t-1}}{P_t}.\qquad \text{(GBC)}$$

(a) Dividing both sides of equation (NFBC) by P_t, and defining $\pi_t=P_t/P_{t-1}-1$ and $n_t=N_t/P_t$, produces

$$m_t+c_t+R_t^{-1}b_{t+1}+q_t s_{t+1}+I_t^{-1}n_t=\tau_t+(q_t+d_t)\,s_t+b_t+\frac{n_{t-1}+m_{t-1}}{1+\pi_t}.\quad \text{(FBC)}$$

(b) Writing the consumer's problem as a Lagrangean, we get

$$\begin{aligned}V(x_t,d_t,\theta_t)=\quad \min_{\lambda_t\geq 0}\ \max_{c_t\geq 0,\ m_t\geq 0,\ s_{t+1},\ b_{t+1},\ n_t}&\Bigg\{\ln(c_t)+\alpha\ln(m_t)\\ &+\lambda_t\left(x_t-m_t-c_t-R_t^{-1}b_{t+1}-q_t s_{t+1}-I_t^{-1}n_t\right)\\ &+\beta E_t V\left([q_{t+1}+d_{t+1}]\,s_{t+1}+b_{t+1}+\frac{n_t+m_t}{1+\pi_{t+1}},d_{t+1},\theta_{t+1}\right)\Bigg\}.\end{aligned}$$

The first order conditions for an interior solution are

$$\begin{aligned}
\frac{1}{c_t} &= \lambda_t, \\
\lambda_t &= \alpha\frac{1}{m_t} + \beta E_t\left(\frac{\partial V[t+1]}{\partial x_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right), \\
\lambda_t q_t &= \beta E_t\left(\frac{\partial V[t+1]}{\partial x_{t+1}}\cdot[q_{t+1}+d_{t+1}]\right), \\
\lambda_t R_t^{-1} &= \beta E_t\left(\frac{\partial V[t+1]}{\partial x_{t+1}}\right), \\
\lambda_t I_t^{-1} &= \beta E_t\left(\frac{\partial V[t+1]}{\partial x_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right).
\end{aligned}$$

Since (following Benveniste-Scheinkman),

$$\frac{\partial V[t]}{\partial x_t} = \lambda_t,$$

these simplify to:

$$\begin{aligned}
q_t\frac{1}{c_t} &= \beta E_t\left(\frac{1}{c_{t+1}}\cdot[q_{t+1}+d_{t+1}]\right), && \text{(EE1)} \\
R_t^{-1}\frac{1}{c_t} &= \beta E_t\left(\frac{1}{c_{t+1}}\right), && \text{(EE2)} \\
I_t^{-1}\frac{1}{c_t} &= \beta E_t\left(\frac{1}{c_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right), && \text{(EE3)} \\
\alpha\frac{1}{m_t} &= \frac{1}{c_t}\left[1-I_t^{-1}\right]. && \text{(CM)}
\end{aligned}$$

(c) Since all consumers are identical, no bonds are traded in equilibrium, so that $b_{t+1} = n_t = 0, \forall t$. Each consumer retains his 1 unit of stock, so that $s_{t+1} = 1, \forall t$. Rewriting the government budget constraint (GBC) in real terms yields

$$\tau_t = m_t - \frac{m_{t-1}}{1+\pi_t}.$$

Inserting this result into equation (FBC), and imposing the equilibrium allocation of stocks and bonds, we get $c_t = d_t$. This allows us to rewrite equations (EE2) and (EE3) as

$$\begin{aligned}
R_t^{-1} &= \beta E_t\left(\frac{d_t}{d_{t+1}}\right), && \text{(EER)} \\
I_t^{-1} &= \beta E_t\left(\frac{d_t}{d_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right). && \text{(EEI)}
\end{aligned}$$

(d) According to the Fisher equation, nominal and real interest rates are related by $I_t = I_{Ft} \equiv R_t E_t(1+\pi_{t+1})$.

1. Using equation (EER), we can rewrite equation (EEI) as

$$\begin{aligned} I_t^{-1} &= \beta E_t\left(\frac{d_t}{d_{t+1}}\right) E_t\left(\frac{1}{1+\pi_{t+1}}\right) + Cov_t\left(\frac{d_t}{d_{t+1}}, \frac{1}{1+\pi_{t+1}}\right), \\ &= R_t^{-1} E_t\left(\frac{1}{1+\pi_{t+1}}\right) + e_{1t}, \\ e_{1t} &\equiv Cov_t\left(\frac{d_t}{d_{t+1}}, \frac{1}{1+\pi_{t+1}}\right) = d_t Cov_t\left(\frac{1}{d_{t+1}}, \frac{1}{1+\pi_{t+1}}\right). \end{aligned}$$

   Note that by Jensen's inequality, we have

$$e_{2t} \equiv E_t\left(\frac{1}{1+\pi_{t+1}}\right) - \frac{1}{E_t\left(1+\pi_{t+1}\right)} > 0.$$

   Insert this result into our equation for I_t^{-1}, and rearrange:

$$\begin{aligned} I_t^{-1} &= R_t^{-1}\left(\frac{1}{E_t\left(1+\pi_{t+1}\right)} + e_{2t}\right) + e_{1t}, \\ \Rightarrow R_t E_t(1+\pi_{t+1}) I_t^{-1} &= 1 + E_t\left(1+\pi_{t+1}\right) e_{2t} + R_t E_t(1+\pi_{t+1}) e_{1t}, \\ \Rightarrow R_t E_t(1+\pi_{t+1}) &= I_t + I_t E_t\left(1+\pi_{t+1}\right) e_{2t} + I_t R_t E_t(1+\pi_{t+1}) e_{1t}, \\ \Rightarrow I_t &= R_t E_t(1+\pi_{t+1}) - I_t E_t\left(1+\pi_{t+1}\right)\left(e_{2t} + R_t e_{1t}\right), \\ &\equiv I_{Ft} + e_t. \end{aligned}$$

2. The deviation of the nominal interest rate I_t from the value implied by the Fisher equation has two components: the approximation error captured by e_{2t}; and the return premium captured by e_{1t}. Let's ignore e_{2t}. Note that when the covariance between the marginal utility of future consumption, $1/d_{t+1}$, and the deflator $1/(1+\pi_{t+1})$ is positive, e_{1t} is positive, e_t is negative, and the nominal rate is below the value implied by the Fisher equation. In this case, inflation is smallest when the marginal utility of consumption is highest, implying that the realized real rate of return on the nominal bond is highest when consumption is low. In short, when e_{1t} is positive, the nominal bond acts like insurance, and investors will accept a lower rate of return. If the covarince between $1/d_{t+1}$ and $1/(1+\pi_{t+1})$ is negative, nominal bonds are "anti-insurance", and investors will demand a return premium.

10. **Cash-in-Advance with home and market goods.**

Consider the following economy:

Time: Discrete; infinite horizon

Demography: Continuum of mass 1 of (representative) households that live for ever.

Preferences: The instantaneous household utility function over, home good consumption, c^h, and market good consumption, c^m, is $u(c^h, c^m)$. $u(.,.)$ is twice differentiable, strictly increasing in both arguments and is strictly concave. We also assume $u_{12}(c^h, c^m) > 0$ and

$$\begin{aligned}\lim_{c^h \to 0} u_1(c^h, c^m) &= \infty \text{ for all } c^m > 0, \\ \lim_{c^m \to 0} u_2(c^h, c^m) &= \infty \text{ for all } c^h > 0,\end{aligned}$$

where u_i represents the derivative of u with respect to the ith argument. These simply mean that household always seek to keep both c^h and c^m positive (no corner solutions). The discount factor is $\beta \in (0, 1)$.

Technology: Each household owns a technology, $f(k)$, which they cannot sell. The technology utilizes capital, k, obtained from foregone consumption to create home goods. Capital completely depreciates in one period (i.e. $\delta = 1$). Capital acquired this period is used for production next period.

Endowments: Households' initial capital stock is k_0. Each household has an initial stock, H_0 of cash.

Institutions: Every period there are markets in which cash is traded for home and market goods. Home goods produced in period t can be consumed (entering the utility function as c_t^h), or they can traded for money. To obtain market goods, c_{t+1}^m, or capital, k_{t+1}, the household has to acquire money this period and trade it for market goods in the next period (cash-in-advance). Market goods are therefore simply someone else's home goods. Market goods acquired in period t can either be consumed (entering the utility function as c_t^m) or they can be converted one-for-one into units of capital, k_t.

There is a government that has the power to levy taxes or make transfers of cash, τ_t, by helicopter drop. They grow the money supply at the rate σ so that the per household stock of cash at time t is given by

$$H_t = H_0 (1 + \sigma)^t.$$

(a) Write down the problem faced by a representative household in period t in recursive form.

In recursive form, the household's problem is

$$V(M_t, k_t) = \max_{c_t^h, c_t^m, k_t, M_t^d} \left\{ u(c_t^h, c_t^m) + \beta V(M_{t+1}, k_{t+1}) \right\}$$

$$\text{subject to:} \quad f(k_t) + \frac{M_t}{p_t} = k_t + c_t^h + c_t^m + \frac{M_t^d}{p_t},$$
$$M_{t+1} = M_t^d + \tau_{t+1},$$
$$\frac{M_t}{p_t} \geq k_{t+1} + c_t^m.$$

where p_t is the period t price of traded goods in terms of cash, M_t^d is nominal money demand, M_t is nominal money holding at the beginning of period t. H_0, k_0, $\{p_t\}_{t=0}^{\infty}$ and $\{\tau_t\}_{t=0}^{\infty}$ are given (perfect foresight).

(b) Derive the necessary conditions for a solution to the household's problem and the envelope conditions.

Write the households problem as a Lagarangian:

$$\begin{aligned} \mathcal{L} &= u(c_t^h, c_t^m) + \beta V(M_t^d + \tau_{t+1}, k_{t+1}) \\ &\quad + \lambda_t \left[f(k_t) + \frac{M_t}{p_t} - k_{t+1} - c_t^h - c_t^m - \frac{M_t^d}{p_t} \right] \\ &\quad + \gamma_t \left[\frac{M_t}{p_t} - k_{t+1} - c_t^m \right]. \end{aligned}$$

The F.O.C's are

$$\begin{aligned} c_t^h &: u_1(c_t^h, c_t^m) - \lambda_t = 0, \\ c_t^m &: u_2(c_t^h, c_t^m) - \lambda_t - \gamma_t = 0, \\ M_t^d &: \beta V_1(M_{t+1}, k_{t+1}) - \frac{\lambda_t}{p_t} = 0, \\ k_{t+1} &: \beta V_2(M_{t+1}, k_{t+1}) - \lambda_t - \gamma_t = 0, \end{aligned}$$

while the complementary slackness condition for the CIA constraint is

$$\gamma_t \left[\frac{M_t}{p_t} - k_{t+1} - c_t^m \right] = 0, \quad \gamma_t \geq 0,$$

and the transversality condition is

$$\lim_{t \to \infty} \beta^t (\lambda_t + \gamma_t) k_t = 0.$$

Finally, the envelope conditions are

$$\begin{aligned} M_t &: V_1(M_t, k_t) = \frac{\lambda_t + \gamma_t}{p_t}, \\ k_t &: V_2(M_t, k_t) = \lambda_t f'(k_t). \end{aligned}$$

(c) Write down the market clearing conditions, the government budget constraint, and define a competitive equilibrium.

The market clearing conditions are

$$\begin{aligned} \text{Money:} \quad & M_t = H_t, \\ \text{Traded Goods:} \quad & k_{t+1} + c_t^m = f(k_t) - c_t^h, \end{aligned}$$

while the government budget constraint is

$$\tau_t = \sigma M_{t-1}.$$

Definition: A perfect foresight competitive equilibrium is an allocation, $\left\{c_t^h, c_t^m, k_t\right\}_{t=0}^{\infty}$, prices, $\{p_t\}_{t=0}^{\infty}$ and sequence of taxes/transfers, $\{\tau_t\}_{t=0}^{\infty}$ such that: given taxes and prices, the allocation solves the household's problem; markets clear; and the government budget constraint holds.

(d) Show that a (strictly) binding cash-in-advance constraint means that households cannot consume efficient amounts of the two goods. Explain your answer.

Combining the first two FOC's from part (b) shows that

$$u_2(c_t^h, c_t^m) = u_1(c_t^h, c_t^m) + \gamma_t,$$

where γ_t is the multiplier on the CIA constraint. It immediately follows that if the CIA constraint strictly binds, $u_1(c_t^h, c_t^m) < u_2(c_t^h, c_t^m)$. Given that money is costless to supply, consumers would better off if marginal utilities were equal.

Combining the first three FOC's with the envelope condition for money shows that

$$u_1(c_t^h, c_t^m) = \beta u_2(c_{t+1}^h, c_{t+1}^m) \frac{1}{1 + \pi_{t+1}},$$

where π_{t+1} is the rate of inflation. This shows that the requirement to go through cash combined with inflation acts like a tax on the consumption of market goods. In equilibrium, $\pi_{t+1} = \sigma$, and consumption of each good is constant. The distortions caused by the CIA constraint will vanish if money growth follows the Friedman rule, i.e., $1 + \sigma = \beta$.