Macroeconomics

Section 1. (Suggested Time: 45 Minutes) *For 3 of the following 6 statements*, state whether the statement is **true**, **false**, or **uncertain**, and give a complete and convincing explanation of your answer. **Note:** Such explanations typically appeal to specific macroeconomic models.

1. Money-in-the-utility-function is a superior modelling technique to cash-in-advance because the model predictions are basically the same but the former approach is more tractable (i.e. easier to analyze).
2. Ricardian equivalence cannot hold in any model that incorporates birth and death (e.g. overlapping generations).
3. Fiscal stimulus increases production.
4. Assuming that government spending is used to produce public goods which increase utility, pro-cyclical government spending is welfare-improving compared to a-cyclical or counter-cyclical government spending.
5. Although the Federal Reserve has dramatically increased the money supply, unemployment in the U.S. remains high. This shows that money is neutral (and superneutral).
6. In a famous article, Robert Shiller found that the prices of stocks are more volatile than the expected present value of the stocks' dividend streams. This shows that stock prices experience bubbles.

Section 2. (Suggested Time: 2 Hours, 15 minutes) Answer any 3 of the following 4 questions.

7. **One-sided search with technological aging**

Time: Discrete, infinite horizon.

Demography: Single worker who lives for ever.

Preferences: The worker is risk-neutral (i.e. $u(x) = x$) and discounts the future at the rate r.

Endowments:

While unemployed she gets a flow utility from leisure of $b > 0$. With probability α she also gets a job offer. All offers are made at the same wage $w > b$ (no need for a distribution function).

Hiring only occurs at a "young" firm. Working for a young firm means that the probability the worker gets laid off in any period is λ_y. In any period, with probability γ, the worker's employer becomes an "old" firm. Working for an old firm means that the probability the worker gets laid off in any period is $\lambda_o > \lambda_y$. (Lay-off occurs due to job destruction so old firms cannot hire unemployed workers.) Assume that events are mutually exclusive so that lay-off by a young firms and the firm getting old cannot happen in the same period and $\lambda_y + \gamma < 1$.

(a) Write down the asset value equations associated with each (of the 3) states the worker can be in. Explain where each comes from.

The Bellman equations are

$$
\begin{aligned}
V_u &= \frac{1}{1+r}\left[b + \alpha V_y + (1-\alpha)V_u\right], \\
V_y &= \frac{1}{1+r}\left[w + \lambda_y V_u + \gamma V_o + (1-\lambda_y-\gamma)V_y\right], \\
V_o &= \frac{1}{1+r}\left[w + \lambda_o V_u + (1-\lambda_o)V_o\right].
\end{aligned}
$$

where V_u is the value to unemployment, V_y is the value to employment at a young firm and V_o is the value to employment at an old firm.

The first equation reflects the fact that the unemployed worker gets b regardless. With probability α she switches (next period) to employment at a young firm. With probability $(1-\alpha)$ she remains unemployed. The second equation follows because the employed worker gets w for sure this period but switches to unemployment with probability λ_y or her firm switches to an old firm with probability γ. With the remaining probability she stays employed at the young firm. The third equation states that someone employed at an old firm gets w this period but switches to unemployment with probability λ_o.

(b) Solve for the **flow** value equations and show that the value to employment at a young firm always exceeds the value to employment at an old firm.

The flow value equations are

$$\begin{aligned} rV_u &= b + \alpha(V_y - V_u), \\ rV_y &= w + \lambda_y(V_u - V_y) + \gamma(V_o - V_y), \\ rV_o &= w + \lambda_o(V_u - V_o). \end{aligned}$$

Subtracting the third equation from the second yields

$$\begin{aligned} (r+\gamma)\left(V_y - V_o\right) &= \lambda_y(V_u - V_y) - \lambda_o(V_u - V_o) \\ \Rightarrow (r+\gamma+\lambda_o)\left(V_y - V_o\right) &= \left(\lambda_o - \lambda_y\right)\left(V_y - V_u\right), \end{aligned} \quad (\dagger)$$

while subtracting the first equation from the second produces

$$(r+\alpha+\lambda_y)\left(V_y - V_u\right) = w - b - \gamma\left(V_y - V_o\right). \quad (*)$$

Combining equations (†) and (*) shows that

$$\left[\frac{(r+\alpha+\lambda_y)\left(r+\gamma+\lambda_o\right)}{\lambda_o - \lambda_y} + \gamma\right]\left(V_y - V_o\right) = w - b > 0,$$

as $w > b$ and $\lambda_o > \lambda_y$.

(c) Despite the result from part b, show that the worker will not prefer to quit to unemployment when her firm becomes old. Explain why not.

Subtract the first flow equation from the third to get

$$(r+\lambda_o)\left(V_o - V_u\right) = w - b - \alpha\left(V_y - V_u\right).$$

Substituting from equation (*) implies

$$(r+\lambda_o)\left(r+\alpha+\lambda_y\right)\left(V_o - V_u\right) = (r+\lambda_y)\left(w-b\right) + \alpha\gamma\left(V_y - V_o\right) > 0,$$

given our results in part (b). One might hypothesize that to take advantage of the possibility of getting a new job with a low lay-off rate a worker might prefer unemployment to employment at an old firm. In which case she would quit her job. This hypothesis turns out to be erroneous. The value to employment at an old firm already takes account of the value to any future spell of unemployment. The worker might as well stay employed, earning w, and wait to get laid-off by the firm.

(d) Now suppose there is a continuum (mass 1) of such workers. Draw the flow diagram associated with movements between each state. Write down a set of equations that can be used to solve for the steady-state numbers (masses) of workers in each state. (You do not need to solve them)

Let u be the mass of unemployed workers and (e_y, e_o) be the mass of workers employed at young and old firms respectively. Figure 1 represents the flow rate diagram.

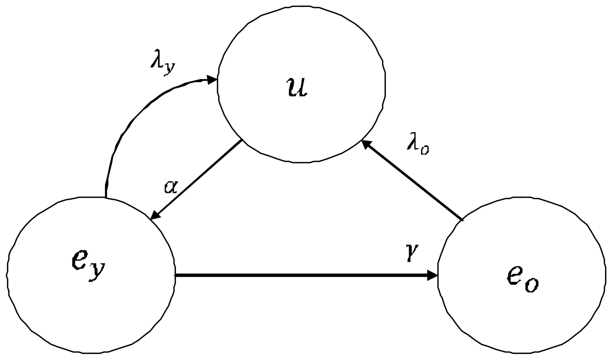

Figure 1

The set of equations required to pin down steady-state are any 3 from the following:

$$\begin{aligned} \alpha u &= \lambda_y e_y + \lambda_o e_o, \\ \gamma e_y &= \lambda_o e_o, \\ \alpha u &= (\lambda_y + \gamma) e_y, \\ u + e_y + e_o &= 1. \end{aligned}$$

8. We are considering a version of the Sidrauski model with endogenous labor supply. The economy is populated by a large number of identical yeoman farmers. Each farmer derives utility from consumption, leisure and real balances, according to:

$$E_0 \left(\sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) - \chi \frac{1}{1+\gamma} (\ell_t - \zeta m_t)^{1+\gamma} \right] \right), \quad \beta, \gamma, \zeta, \chi > 0, \quad \beta < 1.$$

The farmer's resources evolve according to

$$M_t + P_t c_t = P_t(\tau_t + y_t) + M_{t-1}, \tag{NFBC}$$

$$y_t = \ell_t^{\alpha}, \quad 0 < \alpha < 1, \tag{PRF}$$

Note that there is no capital in this economy.

The nominal money supply M_t evolves according to

$$M_t = (1 + \theta_t) M_{t-1}. \tag{MS}$$

In equilibrium, lump-sum taxes equal seigniorage

$$\tau_t = \frac{M_t - M_{t-1}}{P_t}. \tag{GBC}$$

The farmer, however, takes τ_t as given.

(a) The principal benefit to holding money is that it facilitates transactions. This benefit can be thought of as a reduction in time spent on non-leisure activities, or, equivalently, an increase in leisure time. With that interpretation, we can model money as reducing the disutility of labor, as we have above.

(b) Dividing both sides of equation (NFBC) by P_t, and inserting equation (PRF), we get

$$m_t + c_t = \tau_t + \ell_t^\alpha + \frac{m_{t-1}}{1+\pi_t}. \tag{FBC}$$

Write the yeoman farmer's problem as a Lagrangean:

$$E_0\left(\sum_{t=0}^{\infty}\beta^t\left[\ln(c_t) - \chi\frac{1}{1+\gamma}(\ell_t - \zeta m_t)^{1+\gamma} + \mu_t\left(\tau_t + \ell_t^\alpha + \frac{m_{t-1}}{1+\pi_t} - m_t - c_t\right)\right]\right).$$

The first-order conditions are

$$\mu_t = \frac{1}{c_t}, \tag{FOC1}$$

$$\mu_t = \chi(\ell_t - \zeta m_t)^\gamma \zeta + \beta E_t\left(\mu_{t+1}\frac{1}{1+\pi_{t+1}}\right), \tag{FOC2}$$

$$\mu_t \alpha \ell_t^{\alpha-1} = \chi(\ell_t - \zeta m_t)^\gamma. \tag{FOC3}$$

We are assuming that $0 < m_t < \ell_t/\zeta$.

(c) Combining equations (FOC1) and (FOC3) yields the time allocation condition

$$\chi(\ell_t - \zeta m_t)^\gamma = \frac{1}{c_t}\alpha\ell_t^{\alpha-1}. \tag{LL}$$

Combining equations (FOC1) and (FOC2) yields the Euler Equation for money,

$$\frac{1}{c_t} = \zeta\chi(\ell_t - \zeta m_t)^\gamma + \beta E_t\left(\frac{1}{c_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right). \tag{EEM}$$

Finally, equation (GBC) can be written as

$$\tau_t = m_t - \frac{m_{t-1}}{1+\pi_t}.$$

Inserting this results into equation (FBC) yields the resource constraint:

$$y_t = \ell_t^\alpha = c_t. \tag{RC}$$

Combining equations (LL) and (RC) yields

$$\chi\left(c_t^{1/\alpha} - \zeta m_t\right)^\gamma = \frac{1}{c_t}\alpha\left(c_t^{1/\alpha}\right)^{\alpha-1} = \alpha c_t^{-1/\alpha}, \tag{LL$'$}$$

which can also be written as

$$c_t^{1/\alpha} - \left(\frac{\alpha}{\chi}\right)^{1/\gamma} c_t^{-1/(\gamma\alpha)} = \zeta m_t. \tag{LL$''$}$$

(d) Equation (LL$'$) can be written as

$$\gamma \ln\left(\exp\left[\frac{1}{\alpha}\ln(c_t)\right] - \zeta \exp\left[\ln(m_t)\right]\right) = \ln\left(\frac{\alpha}{\chi}\right) - \frac{1}{\alpha}\ln(c_t).$$

Implicitly differentiating, we get

$$\begin{aligned} -\frac{1}{\alpha} d\ln c_t &= \frac{\gamma}{\exp\left(\frac{1}{\alpha}\ln(c_t)\right) - \zeta\exp\left(\ln(m_t)\right)} \\ &\quad \times \left[\frac{1}{\alpha}\exp\left(\frac{1}{\alpha}\ln(c_t)\right) d\ln c_t - \zeta\exp\left(\ln(m_t)\right) d\ln m_t\right], \end{aligned}$$

Let letters with carats "$\widehat{\ }$" denote deviations of logged variables around their steady state values. The previous equation becomes

$$\begin{aligned} -\frac{1}{\alpha}\widehat{c}_t &\approx \frac{\gamma}{c_{ss}^{1/\alpha} - \zeta m_{ss}}\left[\frac{1}{\alpha}c_{ss}^{1/\alpha}\widehat{c}_t - \zeta m_{ss}\widehat{m}_t\right] \\ \Rightarrow \zeta m_{ss}\widehat{m}_t &\approx \left[\frac{1}{\alpha}c_{ss}^{1/\alpha} + \frac{1}{\gamma\alpha}\left(c_{ss}^{1/\alpha} - \zeta m_{ss}\right)\right]\widehat{c}_t \\ \Rightarrow \widehat{c}_t &\approx \frac{\zeta\gamma\alpha m_{ss}}{(1+\gamma)\, c_{ss}^{1/\alpha} - \zeta m_{ss}}\widehat{m}_t \\ &= \frac{\zeta\gamma\alpha m_{ss}}{\gamma c_{ss}^{1/\alpha} + (\alpha/\chi)^{1/\gamma} c_t^{-1/(\gamma\alpha)}}\widehat{m}_t \\ &\equiv \eta\widehat{m}_t. \end{aligned}$$

The next to last line, which uses equation (LL$''$), shows that η is positive. Because money reduces the disutility of work, an increase in the money supply leads to an increase in labor, which in turn leads to an increase in output. Since output cannot be stored, the end result is that consumption increases.

(e) Let's consider optimal steady-state inflation.

1. Combining equations (LL$'$) and (EEM) yields

$$\frac{1}{c_t} = \zeta\alpha c_t^{-1/\alpha} + \beta E_t\left(\frac{1}{c_{t+1}}\cdot\frac{1}{1+\pi_{t+1}}\right).$$

In a steady-state, this becomes

$$\begin{aligned} 1 &= \zeta\alpha c_{ss}^{(\alpha-1)/\alpha} + \beta E_t\left(\frac{c_{ss}}{c_{ss}}\cdot\frac{1}{1+\pi_{ss}}\right) \\ \Rightarrow \zeta\alpha c_{ss}^{(\alpha-1)/\alpha} &= \frac{i_{ss}}{1+i_{ss}} \\ \Rightarrow c_{ss} &= \left(\zeta\alpha\frac{1+i_{ss}}{i_{ss}}\right)^{\alpha/(1-\alpha)}. \end{aligned}$$

where the second line uses the hint that $\beta^{-1}(1+\pi_{ss}) = 1 + i_{ss}$.

2. The answer to part (i) shows that steady state consumption is maximized when $i_{ss} = 0$. This in turn implies that the optimal value of inflation is $\pi_{ss} = \beta - 1$. This is the Friedman rule. If the inflation rate is sufficiently negative, the nominal interest rate will be zero. This will make money costless to hold, and agents will hold arbitrarily large amounts. In our framework, money can substitute for time indefinitely, and in the (non-sensical) extreme case agents will hold so much money that their time endowment, labor supply, and consumption will be infinite. Since money is costless to produce, this outcome is optimal.

9. **Overlapping generation with patient and impatient individuals**

Time: discrete, infinite horizon

Demography: A mass $N_t = N_0(1+n)^t$ of newborns enter in every period where N_0 is the population of newborns in period 0 and n is the growth rate of the population. Everyone lives for 2 periods except for the first generation of old people. A share ϕ^p of each generation are patient and ϕ^m of the population are impatient where $\phi^p + \phi^m = 1$. (Variables and parameters that relate to patient individuals carry a superscript p those relating to impatient individuals carry a superscript m.)

Preferences: for the generations born in and after period 0;

$$U_t^j(c_{1,t}^j, c_{2,t+1}^j) = u(c_{1,t}^j) + \beta^j u(c_{2,t+1}^j), \qquad j = p, m,$$

where $c_{i,t}^j$ is consumption by type j individuals in period t and stage of life i. The common instantaneous utility function $u(.)$ is increasing strictly concave and twice differentiable with $\lim_{x \to 0} u'(x) = \infty$. For the initial old generation $\tilde{U}^j(c_{2,0}^j) = u(c_{2,0}^j)$. The types of individuals therefore differ only by their lifetime discount factors: $\beta^p > \beta^m$.

Technology: $F(K, L)$ is a constant returns to scale neoclassical production technology which uses capital and labor to produce a consumption good. (Capital depreciates 100% in use.) Consumption goods can be consumed or they can be used as the capital input to the production. It will be convenient to express the per (total) young population output as $f(k_t)$ where k_t is the capital stock per young person (there is no need to distinguish who owns the capital).

Endowments: Everyone has one unit of labor services when young. When old no one can work. Everyone among the initial old generation has the same k_0 units of capital which they can rent out to firms. (We can think of a single collectively owned firm which takes wages and interest rates as given. The firm will make no profits in equilibrium.)

(a) Write down the problems faced by each type of individual in the economy and solve for the Euler equations for each type.

$$\max_{s_t^j} \; u(w_t - s_t^j) + \beta^j u(R_{t+1} s_t^j), \qquad j = p, m.$$

Euler equation:

$$-u'(w_t - s_t^j) + \beta^j R_{t+1} u'(R_{t+1} s_t^j) = 0, \qquad j = p, m.$$

(b) Show that the savings of each type can be expressed (implicitly) as a function of the wage and interest rate.

We need to confirm that the Euler equation defines a single-valued function. One way to do this is to show that the second order condition implies a unique max for given w_t and R_{t+1}:

$$u''(w_t - s_t^j) + \beta^j R_{t+1}^2 u''(R_{t+1} s_t^j) < 0.$$

Equivalently, notice that as s_t^j approaches w_t the LHS of the Euler condition approaches $+\infty$, while as s_t^j approaches 0 the LHS of the Euler condition approaches $-\infty$.

(c) Write down and solve the firm's problem to obtain the wage and interest rate as a function of the average capital holding per young person.

Firm's problem

$$\max_{k_t} \; f(k_t) - w_t - R_t k_t.$$

Solving this implies

$$\begin{aligned} R_t &= R(k_t) = f'(k_t), \\ w_t &= w(k_t) = f(k_t) - R_t k_t. \end{aligned}$$

(d) Write down the market clearing condition for capital (in terms of the saving functions) and define a competitive equilibrium. Write down an (implicit) expression for the steady state equilibrium capital stock, k^*.

Market clearing.

$$(1+n)k_{t+1} = \phi^p s^p(w(k_t), R(k_{t+1})) + \phi^m s^m(w(k_t), R(k_{t+1})).$$

A competitive equilibrium is an allocation $\{c_{1,t}^j, c_{2,t+1}^j\}_{t=0}^{\infty}$ $j = p, m$, a sequence of capital stocks $\{k_t\}_{t=1}^{\infty}$ and a sequence of prices $\{w_t, R_{t+1}\}_{t=0}^{\infty}$ such that, given prices the allocation solves the household problems, the sequence of capital stocks solve the firm's problem at each t; and markets clear.

In steady state, k^* solves

$$(1+n)k^* = \phi^p s^p(w(k^*), R(k^*)) + \phi^m s^m(w(k^*), R(k^*)).$$

(e) Write down the Planner's problem (just write the Lagrangian in terms of k_t if you like) assuming the Planner treats everyone equally. Solve for the efficiency conditions.

The Lagrangian for the planner's problem is

$$\begin{aligned} \mathcal{L} &= \sum_{j \in \{p,m\}} \left\{ u(c_{2\,0}^j) + \sum_{t=0}^{\infty} u(c_{1t}^j) + \beta^j u(c_{2t+1}^j) \right\} \\ &\quad + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) - (1+n)k_{t+1} - \sum_{j \in \{p,m\}} \phi^j \left(c_{1t}^j + \frac{c_{2t}^j}{1+n} \right) \right]. \end{aligned}$$

F.O.C's:

$$\begin{aligned}
c_{20}^{j} &: \quad u'(c_{20}^{j}) - \lambda_0 \frac{\phi^j}{1+n} = 0, \\
c_{1t}^{j} &: \quad u'(c_{1t}^{j}) - \lambda_t \phi^j = 0, \\
c_{2t}^{j} \text{ (for } t > 0) &: \quad \beta^j u'(c_{2t}^{j}) - \lambda_t \frac{\phi^j}{1+n} = 0, \\
k_t &: \quad \lambda_t f'(k_t) - \lambda_{t-1}(1+n) = 0.
\end{aligned}$$

In a steady-state, $\lambda_t = \lambda_{ss}$, $k_t = k_{ss}$, etc. Ignoring the initial old, the optimality condition for the Planner's allocation for both types is

$$u'(c_{1ss}^{j}) = \beta^j (1+n) u'(c_{2ss}^{j}),$$

while the associated capital stock, k_{ss} solves $f'(k_{ss}) = 1 + n$.

(f) Under what circumstances would a steady-state equilibrium in the market economy represent a Pareto optimum?

Ignoring the current old, the market equilibrium will be efficient if $k^* = k_{ss}$. If $f'(k^*) > 1 + n$ then $k^* < k_{ss}$. Increasing k^* to k_{ss} would require taking income from the initial old (or some other generation), so in this case the market economy achieves a Pareto optimum. If $f'(k^*) < 1+n$ then $k^* > k_{ss}$. In this case the capital stock could be reduced to k_{ss} by giving someone some extra consumption in the short term. In this case the market economy fails to achieve a Pareto optimum.

(g) Comment on the distinction between this economy and one with a homogeneous population.

Introducing two types of individual makes no difference to the efficiency properties of the steady-state competitive equilibrium. The patient households will consume more in old age than the impatient households but each type chooses their consumption pattern optimally given the interest rate they both face.

10. Consider an economy populated by a large number of identical yeoman farmers. Each farmer's preferences over consumption and leisure are:

$$E_t \left(\sum_{j=0}^{\infty} \beta^j \frac{1}{1-\gamma} c_{t+j}^{1-\gamma} (1 - \ell_{t+j}) \right), \quad 0 < \beta < 1, \quad 0 \le \gamma < 1.$$

Let R_t^{-1} be the price of a risk-free discount bond that pays one unit of consumption at time $t+1$, and let b_{t+1} denote the quantity of such bonds. The farmer's resources evolve according to

$$R_t^{-1} b_{t+1} + c_t = d_t \ell_t^{1-\alpha} + b_t, \tag{FBC}$$

Output depends on the exogenous shifter d_t, where d_t is a non-negative random variable governed by a Markov process with the stationary transition density $f(d', d)$. Output is not storable—there is no capital.

Suppose for now that d_t is common to all farmers.

(a) Noting that $R_t^{-1} = R_t^{-1}(d_t)$, we write the consumer's problem as a Lagrangean:

$$V(b_t, d_t) =$$
$$\min_{\lambda_t \geq 0} \max_{c_t \geq 0, \ell_t \in [0,1], b_{t+1}} \frac{1}{1-\gamma} c_t^{1-\gamma}(1-\ell_t) + \lambda_t \left(b_t + d_t \ell_t^{1-\alpha} - c_t - R_t^{-1} b_{t+1}\right)$$
$$+\beta \int V(b_{t+1}, d_{t+1}) \times f(d_{t+1}, d_t)\, dd_{t+1},$$

where $f(\cdot,\cdot)$ gives the conditional density of d_{t+1}. The FOC for an interior solution are:

$$c_t^{-\gamma}(1-\ell_t) = \lambda_t, \tag{FOC1}$$
$$\frac{1}{1-\gamma} c_t^{1-\gamma} = \lambda_t (1-\alpha) d_t \ell_t^{-\alpha}, \tag{FOC2}$$
$$\lambda_t R_t^{-1} = \beta \int \frac{\partial V[t+1]}{\partial b_{t+1}} f(d_{t+1}, d_t)\, dd_{t+1}. \tag{FOC3}$$

Since (following Benveniste-Scheinkman),

$$\frac{\partial V[t]}{\partial b_t} = \lambda_t,$$

the Euler equation for bonds is

$$R_t^{-1} c_t^{-\gamma}(1-\ell_t) = \beta E_t\left(c_{t+1}^{-\gamma}(1-\ell_{t+1})\right). \tag{EE}$$

Combining equations (FOC1) and (FOC2) yields the labor allocation condition:

$$\frac{1}{1-\gamma} c_t = (1-\ell_t)(1-\alpha) d_t \ell_t^{-\alpha}. \tag{LL}$$

(b) Next we find equilibrium bond prices.

1. With identical agents and non-storable goods, equilibrium bond holdings are identically zero.
2. With bond holdings set to zero, equation (FBC) becomes: $c_t = d_t \ell_t^{1-\alpha}$. Inserting this result into equation (LL) yields

$$\frac{1}{1-\gamma} d_t \ell_t^{1-\alpha} = (1-\ell_t)(1-\alpha) d_t \ell_t^{-\alpha}$$
$$\Rightarrow \frac{1}{(1-\gamma)(1-\alpha)} \ell_t = 1-\ell_t$$
$$\Rightarrow \ell_t = \frac{(1-\gamma)(1-\alpha)}{(1-\gamma)(1-\alpha)+1} \equiv \ell_e.$$

With our preference specification and a Cobb-Douglas production function, the income and substitution effects of a productivity change offset exactly, leaving labor constant.

3. With this result, $c_t = d_t \ell_e^{1-\alpha}$, and equation (EE) becomes

$$\begin{aligned} R_t^{-1} &= \beta E_t \left(\frac{c_{t+1}^{-\gamma}(1-\ell_{t+1})}{c_t^{-\gamma}(1-\ell_t)} \right) \\ &= \beta E_t \left(\frac{\left[d_{t+1}\ell_e^{1-\alpha}\right]^{-\gamma}(1-\ell_e)}{\left[d_t\ell_e^{1-\alpha}\right]^{-\gamma}(1-\ell_e)} \right) \\ &= \beta E_t \left(\left(\frac{d_t}{d_{t+1}} \right)^{\gamma} \right) \\ &\equiv R_t^{-1}(d_t), \end{aligned}$$

with the last line utilizing the fact that $\{d_t\}$ is Markov.

(c) Now suppose there are two types of farms, occuring in equal proportions. The productivity process for wet-weather farms is

$$d_t^W = \delta + d_t,$$

while productivity for arid- (dry) weather farms follows

$$d_t^A = \delta - d_t.$$

d_t still follows a Markov process with a stationary transition density, but is now zero-mean, and bounded by $-\delta < d_t < \delta$.

1. Let c^W (c^A) and ℓ^W (ℓ^A) denote the consumption and labor of a wet-weather (arid-weather) farmer. Given equal weights, the social planner's problem can be written as

$$\max_{\{c_t^W, c_t^A, \ell_t^W, \ell_t^A,\}} E_0 \left(\sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\gamma} \left(c_t^W\right)^{1-\gamma} \left(1-\ell_t^W\right) + \frac{1}{1-\gamma} \left(c_t^A\right)^{1-\gamma} \left(1-\ell_t^A\right) \right] \right),$$

$$s.t. \quad c_t^W + c_t^A = d_t^W \left(\ell_t^W\right)^{1-\alpha} + d_t^A \left(\ell_t^A\right)^{1-\alpha}, \tag{RC}$$

$$1 \leq \ell_t^W \leq 1; \quad 1 \leq \ell_t^A \leq 1,$$

2. With no storage, the social planner's problem boils down to a series of static problems. Dropping the time subscript, the first-order conditions for any period are

$$\begin{aligned} \left(c^W\right)^{-\gamma}\left(1-\ell^W\right) &= \mu, \\ \left(c^A\right)^{-\gamma}\left(1-\ell^A\right) &= \mu, \\ \frac{1}{1-\gamma}\left(c^W\right)^{1-\gamma} &= \mu d^W (1-\alpha) \left(\ell^W\right)^{-\alpha} \\ \frac{1}{1-\gamma}\left(c^A\right)^{1-\gamma} &= \mu d^A (1-\alpha) \left(\ell^A\right)^{-\alpha}, \end{aligned}$$

where μ denotes the multiplier on the resource constraint. Eliminating the multiplier yields

$$\left(c^W\right)^{-\gamma}\left(1-\ell^W\right) = \left(c^A\right)^{-\gamma}\left(1-\ell^A\right), \tag{SP1}$$

$$\frac{1}{1-\gamma} c^W = d^W \left(1-\ell^W\right)(1-\alpha)\left(\ell^W\right)^{-\alpha}, \tag{SP2}$$

$$\frac{1}{1-\gamma} c^A = d^A \left(1-\ell^A\right)(1-\alpha)\left(\ell^A\right)^{-\alpha}. \tag{SP3}$$

These three equations, along with the resource constraint (RC), characterize the solution to the social planner's problem.

3. Equation (SP1) implies that

$$\frac{c^W}{c^A} = \left(\frac{1-\ell^W}{1-\ell^A}\right)^{1/\gamma}.$$

Equations (SP2) and (SP3) imply that

$$\frac{c^W}{c^A} = \left(\frac{d^W \left(\ell^W\right)^{-\alpha}}{d^A \left(\ell^A\right)^{-\alpha}}\right)\left(\frac{1-\ell^W}{1-\ell^A}\right).$$

Put together, we get

$$\left(\frac{1-\ell^W}{1-\ell^A}\right)^{(1-\gamma)/\gamma}\left(\frac{\ell^W}{\ell^A}\right)^{\alpha} = \frac{d^W}{d^A},$$

which shows that ℓ^W and ℓ^A vary with d^W and d^A. Once we allow productivity to vary across groups, the social planner will begin to transfer consumption between groups as well. This breaks the link between labor hours and consumption for each farmer, so that the income and substitution effects of productivity changes no longer cancel exactly, and hours vary.