1. Borrowing and Lending in a 2-Period Model

Consider consumers who live for two periods. Their utility function is $U_1(c_1, c_2) = \log(c_1) + \frac{1}{1+\rho_1}\log(c_2)$ and they receive an exogenous stream of income every period denoted by y_1 and y_2, respectively. In the first period of their lives, consumers decide how much to consume and how much to save or borrow. The consumers can borrow and lend at an interest rate, r.

a. Write down the intertemporal budget constraint faced by the consumer.

$$(1+r)y_1 + y_2 = (1+r)c_1 + c_2$$

b. Solve the maximization problem and determine the consumer's consumption in the first period.

$$\max\left\{\log(c_1) + \frac{1}{1+\rho_1}\log(c_2)\right\} - \lambda[(1+r)y_1 + y_2 - (1+r)c_1 + c_2]$$

$$FOC[c_1]: \frac{1}{c_1} = (1+r)\lambda$$

$$FOC[c_2]: \frac{1}{(1+\rho_1)c_2} = \lambda$$

Eliminate λ to get

$$\frac{c_2}{c_1} = \frac{1+r}{1+\rho_1}$$

Plug the FOC into the budget constraint and solve:

$$(1+r)y_1 + y_2 = (1+r)c_1 + c_1\frac{1+r}{1+\rho_1}$$

$$c_1 = \frac{[(1+r)y_1 + y_2](1+\rho_1)}{(2+\rho_1)(1+r)}.$$

c. Determine when a consumer is going to be a saver or borrower as a function of the interest rate and his or her own discount factor.

If the consumer is a saver, then his or her income in the first period exceeds his or her consumption in that period:

$$y_1 - c_1 \geq 0$$

$$y_1 - \frac{[(1+r)y_1 + y_2](1+\rho_1)}{(2+\rho_1)(1+r)} \geq 0.$$

$$\frac{1+r}{1+\rho_1} \geq \frac{y_2}{y_1}$$

Now assume there is a second type of consumer in the economy who only differs in his or her individual discount factor such that his utility function is $U_2(c_1, c_2) = \log(c_1) + \frac{1}{1+\rho_2}\log(c_2)$, where $\rho_1 \geq \rho_2$. There is an equal number of consumers of each type in the economy. Also assume that net savings are zero in aggregate. In other words, there is no outside supply of deposits or bonds, and the consumers have to borrow from each other and lend to each other.

d. Determine the equilibrium interest rate. *Hint:* In equilibrium, the amount borrowed by the borrowers must equal the amount saved by the savers.

Since $\rho_1 \geq \rho_2$, we know that type two consumer is more patient (he or she discounts the second-period consumption less) and therefore is going to lend to type one consumer, who is going to borrow. As there is an equal number of each type in the economy, we can simply equate the saving of type two to the borrowing of type one:

$$y_1 - c_1^2 = c_1^1 - y_1$$
$$y_1 - \frac{[(1+r)y_1 + y_2](1+\rho_2)}{(2+\rho_2)(1+r)} = \frac{[(1+r)y_1 + y_2](1+\rho_1)}{(2+\rho_1)(1+r)} - y_1$$
$$r = \frac{y_2}{y_1}\frac{4+3\rho_1+3\rho_2+2\rho_1\rho_2}{4+\rho_1+\rho_2}$$

e. Verify, using the condition you derived in part c and the interest rate you derived in part d that indeed one type of consumer is going to be a borrower and the other type a saver.

We can check that type one is really a borrower:

$$y_1 - c_1^1 \leq 0$$
$$y_1 - \frac{[(1+r)y_1 + y_2](1+\rho_1)}{(2+\rho_1)(1+r)} \leq 0$$
$$y_1 - y_2\frac{1+\rho_1}{1+r} \leq 0$$
$$y_1 - y_2(1+\rho_1)\frac{y_1}{y_2}\frac{4+\rho_1+\rho_2}{4+3\rho_1+3\rho_2+2\rho_1\rho_2} \leq 0$$
$$(2+\rho_1)(\rho_2 - \rho_1) \leq 0$$

which implies $\rho_2 \leq \rho_1$. This is the assumption we started with. Similarly, we can check that type two is a saver.

2. Overlapping Generations Model

Consider an economy described by an overlapping generations model. Individuals live for two periods. In the first period of his life each individual works and earns a wage w_t, and decides how much to consume and how much to save and invest at a rate r_{t+1}. In the second period individuals consume all their savings. Assume no depreciation, no population growth, no technological progress and $A = 1$. The individual's utility function is

$U(c_{1t}, c_{2t+1}) = \frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta}$ and the production function in per capita terms is $y_t = k_t^{\alpha}$. Assume markets are competitive and thus labor and capital earn their marginal products.

a. Determine the intertemporal budget constraint of each individual and use it to solve for the first-period consumption and saving.

The budget constraint is $(w_t - c_{1t})(1+r_{t+1}) = c_{2t+1}$. Therefore the problem becomes

$$\max\left\{\frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta}\right\} - \lambda[(w_t - c_{1t})(1+r_{t+1}) - c_{2t+1}]$$

$$FOC[c_{1t}]: c_{1t}^{-\theta} = \lambda(1+r_{t+1})$$

$$FOC[c_{2t+1}]: \frac{1}{1+\rho}c_{2t+1}^{-\theta} = \lambda$$

We eliminate λ to get $c_{2t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} c_{1t}$

Combining this with the budget constraint we get $c_{1t} = \frac{(1+\rho)^{1/\theta} w_t}{(1+\rho)^{1/\theta} w_t + (1+r_{r+1})^{(1-\theta)/\theta}}$.

Saving is then $w_t - c_{1t} = w_t = \frac{(1+\rho)^{1/\theta} w_t}{(1+\rho)^{1/\theta} w_t + (1+r_{r+1})^{(1-\theta)/\theta}} = w_t\left[\frac{(1+r_{r+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} w_t + (1+r_{r+1})^{(1-\theta)/\theta}}\right]$.

b. When is the savings rate constant and what form does the utility function then take (i.e. what mathematical function does it equal)?

The saving rate is just the total saving as a fraction of total income, therefore in this case

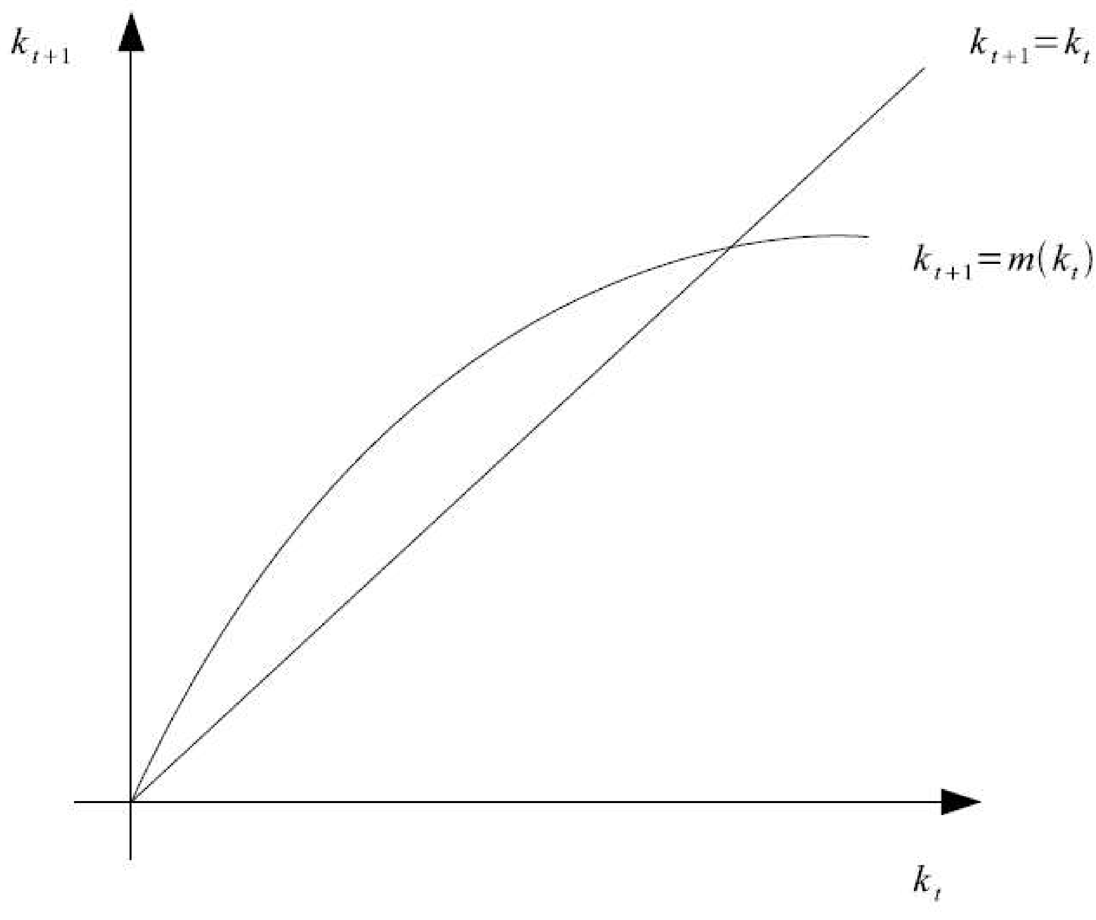
$s(r_{t+1}) = \frac{(1+r_{r+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} w_t + (1+r_{r+1})^{(1-\theta)/\theta}}$. This is constant when $\theta = 1$, which then implies a logarithmic

utility function, i.e. $U(c_{1t}, c_{2t+1}) = \log(c_{1t}) + \frac{1}{1+\rho}\log(c_{2t+1})$.

From now on assume that the saving rate is indeed constant (you do not need to have solved part (b) to continue).

c. Determine the relationship between k_t and k_{t+1}, show it in a graph, and determine the steady-state level of capital per capita, k^*.

Total capital in a given period equals savings of the young generation from the previous period, $K_{t+1} = s(r_{t+1})L_t w_t$. In per capita terms, $k_{t+1} = s(r_{t+1})w_t$. Using the constant saving rate from above and the fact that labor is paid its marginal product, $k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{2+\rho}$. Then in steady state, $k^* = \left(\frac{1-\alpha}{2+\rho}\right)^{\frac{1}{1-\alpha}}$.



Note: in the graph, $k_{t+1} = m(k_t) = \frac{(1-\alpha)k_t^{\alpha}}{2+\rho}$.

d. How is this steady-state level of capital per capita affected by a tax on the interest rate on savings (making the after-tax interest rate $r_{t+1}(1-\tau_{rt+1})$, where τ_{rt+1} is the tax rate)? You do not need to resolve the model to answer this question and you may disregard any implications it may have for government revenue.

The interest rate tax will have no effect on steady-state level of capital because saving rate is constant and thus independent of the interest rate or any tax on it.

e. How is this steady-state level of capital per capita affected by a tax on labor income (making the after-tax income $w_t(1-\tau_{wt})$, where τ_{wt} is the tax rate)? Again, you do not need to resolve the model to answer this question and you may disregard any implications it may have for government revenue.

The labor income tax is going to decrease the steady-state level of capital, because it decreases effective payment to labor and therefore the capital accumulation:

$$k_{t+1} = s(r_{t+1})w_t(1-\tau_{wt}) = \frac{(1-\alpha)(1-\tau_{wt})k_t^{\alpha}}{2+\rho}$$

$$k^* = \left(\frac{(1-\alpha)(1-\tau_w)}{2+\rho}\right)^{\frac{1}{1-\alpha}}$$

assuming that the tax does not change over time.

3. Social Security reform.

Write a short paragraph in answer to each part.

a. What is the problem that current Social Security reform proposals are aiming to fix? Discuss demography as well as economics.

Social Security faces a future deficit when its tax revenues will be smaller than its benefit payments. This deficit will arise from two reasons. First, the baby boomers, born 1945-65, will retire. Second, everyone is living longer. The result is that the ratio of people 20-64 who pay Social Security taxes will fall relative to the number of people >65 who receive Social Security benefits.

b. Summarize the reforms proposed by Diamond and Orszag.

Diamond and Orszag propose to eliminate the deficit without changing the basic structure of Social Security. They propose to raise the tax on wages and to reduce the benefit to retirees. They do this in a variety of ways in order to make the Social Security system more progressive, account for longer lives, and compensate for the greater dispersion of earnings.

c. Summarize the reforms proposed by Feldstein. Explain the difference between the two proposals in terms of a 2x2 matrix of unfunded and funded plans versus defined benefit and defined contribution plans.

Feldstein proposes to eliminate the deficit by introducing Individual Accounts. In terms of the 2x2 matrix, he wants to move diagonally from an unfunded, defined-benefit plan to a funded, defined-contribution plan. To eliminate double taxation of the current young generation Feldstein proposes a voluntary contribution by them to their Individual Accounts. This contribution, matched by an equal diversion to their Individual Accounts from their regular Social Security taxes, will increase the capital stock, according to Feldstein. The income from the added capital will pay for the extra expense of shifting to a funded system.

d. Describe the 1983 Social Security reforms in terms of the 2x2 matrix. Explain briefly how Diamond analyzed this reform. What were his main assumptions and simplifications, and his major conclusions?

The 1983 Social Security reforms increased the payroll tax on wages in order to build up a trust fund to pay for the baby boomers' retirement. In terms of the 2x2 matrix, it moved the Social Security system vertically from an unfunded to a funded, defined-benefit system. He assumed $\theta = 1$ and a simple linear technology with no population growth or technical progress. He introduced α as a measure of the extent to which the rise in payroll taxes was offset by a fall in other, that is, income, taxes. He concluded that α in his model was the critical determinant of the results.

e. Under what circumstances will a move along the diagonal of the matrix (as in part (c) increase the national capital stock? Under what circumstances will it fail to do so?

Diamond showed that if $\alpha = 0$ and there is no change in other taxes, then funding the Social Security system will increase national capital. Feldstein assumed this will happen. But if $\alpha = 1$ and income taxes are reduced when payroll taxes are raised, there is a decrease in national capital. In order to know the effects of changes in Social Security, we need to know how the government will respond to these changes.