1. Determine the derivatives of the following functions

   (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.
   Solution: $f'(x) = \frac{4x}{(x^2+1)^2}$.

   (b) $f(x) = \left(x^2 + 1\right)\sqrt{x^2 - 1}$.
   Solution: $f'(x) = \frac{x(3x^2-1)}{\sqrt{x^2-1}}$.

2. Determine the derivatives $\frac{dy}{dx}$ of the following functions at the given points

   (a) $y(x)$ as a function of x defined by the equation

   $$x^3 + y^3 = 2,$$

   at point $(1, 1)$.
   Solution: $\frac{dy}{dx} = -\frac{x^2}{y^2} = -1.$

   (b) $y(x)$ as a function of x defined by the equation

   $$xy^3 + x^2y^2 + x^3y = -1,$$

   at point $(-1, 1)$.
   Solution: $\frac{dy}{dx} = -\frac{y^3 + 2xy^2 + 3x^2y}{3xy^2 + 2x^2y + x^3} = 1.$

3. Consider the composition of three functions

   (a) Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions. Derive a formula for the derivative of the function $f(g(h(x)))$.
   Solution:
   $$\frac{d}{dx}\left(f(g(h(x)))\right) = f'(g(h(x)))g'(h(x))h'(x).$$

   (b) Write the function $\sqrt{1+\sqrt{1+x^2}}$ in the form $f(g(h(x)))$, and use your formula from above to compute the derivative of this function.
   Solution: Let $f(x) = \sqrt{1+x}$, $g(x) = \sqrt{1+x}$ and $h(x) = x^2$.
   $$\frac{d}{dx}\left(f(g(h(x)))\right) = \frac{x}{2\sqrt{1+x^2}\sqrt{1+\sqrt{1+x^2}}}.$$

4. Let $f(x) = (2x^2+3)^{3/2}$. Show that $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Sketch the graph of this function and find the global minimum of $f(x)$ for $-\infty < x < \infty$.
Solution:
$$f'(x) = 6x\sqrt{2x^2+3}, \quad f''(x) = \frac{6(4x^2+3)}{\sqrt{2x^2+3}}.$$
The factor $\sqrt{2x^2+3}$ in $f'(x)$ is always positive. Hence $f'(x) < 0$ is negative for $x < 0$ and positive for $x > 0$. Therefore $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Since $f''(x)$ is always positive, the graph is concave up, without inflection points. The minimum is at $x = 0$.

5. If the demand equation for a monopolist is $p = 150 - 0.02x$ and the cost function is $C(x) = 10x + 300$, find the value of x that maximizes the profit.
Solution: The profit

$$P(x) = xp - C(x) = x\,(150 - 0.02x) - (10x + 300) = 140x - 0.02x^2 - 300,$$

which is a quadratic with negative second order coefficient -0.02. Hence the graph is concave down, with max at

$$P'(x) = 140 - 0.04x = 0,$$

or $x = 3500$.

6. An open rectangular box is to be 4 feet long and have a volume of 200 cubic feet. Find the dimensions for which the amount of material needed to construct the box is as small as possible.
   Solution: Let the height be x feet and width y feet. Then the constraint is $4xy = 200$ or $xy = 50$. Since the box is open, one only needs to construct the four sides and the bottom of the box. Hence the objective to b e minimized is $4y + 2(xy + 4x) = 2xy + 4y + 8x$. Replace y by $50/x$, the objective is

$$C(x) = 2x * 50/x + 4 * 50/x + 8x = 100 + 200/x + 8x.$$

   Letting

$$C'(x) = -200/x^2 + 8 = 0,$$

   we have $x = 5$ and hence $y = 10$.