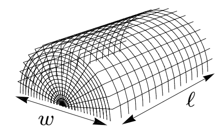
1. [15 points] A hoophouse is an unheated greenhouse used to grow certain types of vegetables during the harsh Michigan winter. A typical hoophouse has a semi-cylindrical roof with a semi-circular wall on each end (see figure to the right). The growing area of the hoophouse is the rectangle of length ℓ and width w (each measured in feet) which is covered by the hoophouse. The cost of the semi-circular walls is \$0.50 per square foot and the cost of the roof, which varies with the side length ℓ, is $\$1 + 0.001\ell$ per square foot.



a. [4 points] Write an equation for the cost of a hoophouse in terms of ℓ and w. *(Hint: The surface area of a cylinder of height ℓ and radius r, not including the circles on each end, is $A = 2\pi r\ell$.)*

Solution: The roof has area $\pi r\ell = \frac{\pi}{2}w\ell$. The walls have area $\pi r^2 = \frac{\pi}{4}w^2$. This means the cost is

$$C = 0.50 \cdot \frac{\pi}{4}w^2 + (1 + 0.001\ell)\frac{\pi}{2}w\ell = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001\ell)w\ell.$$

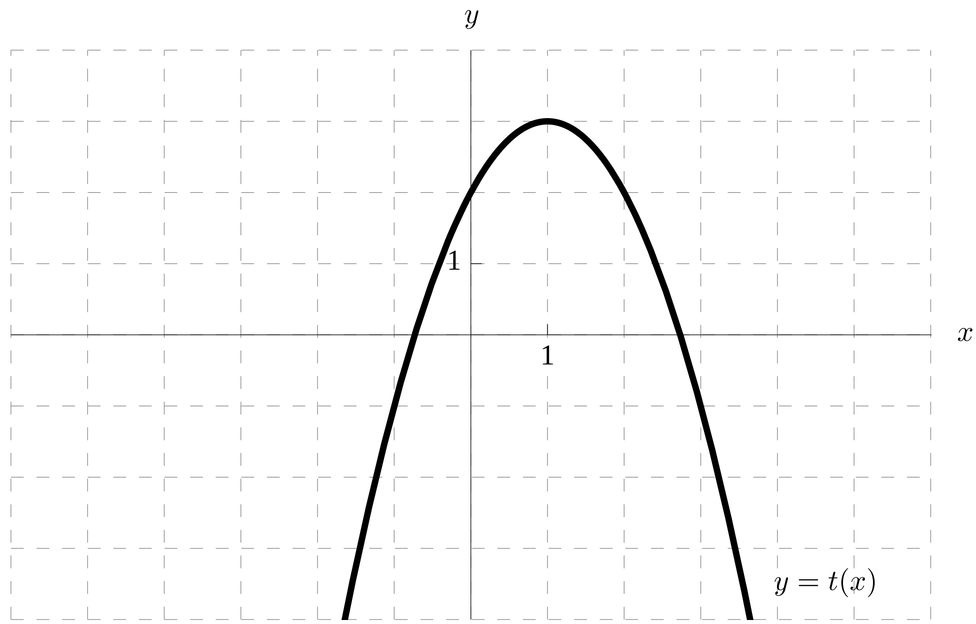
b. [11 points] Find the dimensions of the least expensive hoophouse with 8000 square feet of growing area.

Solution: The Area of the hoophouse is $8000 = w\ell$. Using this expression, we can eliminate ℓ in our cost equation.

$$C = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001\ell)w\ell = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001(8000/w))8000.$$
$$= 4000\pi + \frac{\pi}{8}w^2 + 32000\pi w^{-1}.$$

Now we compute $C' = \frac{\pi}{4}w - 32000\pi w^{-2}$. Solving for w gives us a critical point at $w =$ 50.397ft. To see what type of critical point we have, we compute $C'' = \frac{\pi}{4} + 64000\pi w^{-3}$. For $w > 0$ $C'' > 0$ which means our critical point is a local minimum by the second derivative test. Since it is the only critical point of the function, it must be a global minimum as well. When $w = 50.397$, $\ell = 158.74$, so the least expensive hoophouse with 8000 square feet of growing area is 50.397 x 158.74 ft.

2. [16 points]
Graphed below is a function $t(x)$. Define $p(x) = x^2 t(x)$, $q(x) = t(\sin(x))$, $r(x) = \frac{t(x)}{3x+1}$, and $s(x) = t(t(x))$. For this problem, do not assume $t(x)$ is quadratic.



Carefully estimate the following quantities.

a. [4 points] $p'(-1)$

Solution: By the product rule, $p'(x) = 2xt(x) + x^2t'(x)$. Estimating using the graph, we have

$$p'(-1) = 2(-1)t(-1) + (-1)^2t'(-1) = (-2)(-1) + 4 = 6.$$

b. [4 points] $q'(0)$

Solution: By the chain rule, $q'(x) = t'(\sin x)\cos x$. Estimating using the graph, we have

$$q'(0) = t'(\sin 0)\cos 0 = t'(0) = 2.$$

c. [4 points] $r'(3)$

Solution: By the quotient rule, $r'(x) = \frac{(3x+1)t'(x)-3t(x)}{(3x+1)^2}$. Estimating using the graph, we have

$$r'(3) = \frac{(3(3)+1)t'(3) - 3t(3)}{(3(3)+1)^2} = \frac{-40-3(-1)}{100} = -\frac{37}{100}$$

d. [4 points] $s'(0)$

Solution: By the chain rule, $s'(x) = t'(t(x))t'(x)$. Estimating using the graph, we have

$$s'(0) = t'(t(0))t'(0) = t'(2)\cdot 2 = (-2)(2) = -4.$$

3. [12 points] Representative values of the derivative of a function $f(x)$ are shown in the table below. Assume $f'(x)$ is a continuous function and that the values in the table are representative of the behavior of $f'(x)$.

x	0	0.5	1	1.5	2	2.5	3
$f'(x)$	1	0.3	0	-0.1	-0.15	-0.12	-0.10

a. [6 points] Estimate the location of the global maximum and minimum of $f(x)$ on the closed interval $[0, 3]$. Justify your answers based on the data in the table.

Solution: We note that $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. Thus $f(x)$ has a local maximum at $x = 1$. Further, because there is only one change of sign in the derivative, we know that this is the global maximum. The global minimum will occur at one of the endpoints. It is not easy to tell at which endpoint this occurs, but because the negative slopes are of smaller magnitude (for $x > 1$) than the positive slopes (for $x < 1$), we expect that the global minimum occurs at $x = 0$.

b. [6 points] Can you tell from these data if $f(x)$ has any inflection points? If so, estimate the location of any inflection points and indicate how you know their locations. If not, explain why not.

Solution: We know that an inflection point occurs when $f'(x)$ goes from increasing to decreasing or vice versa. We can see from these data that $f'(x)$ is decreasing until sometime between $x = 2$ and $x = 2.5$, and increasing thereafter. Thus there is an inflection point at an x somewhere in $(2, 2.5)$.

4. [15 points] A model for the amount of an antihistamine in the bloodstream after a patient takes a dose of the drug gives the amount, a, as a function of time, t, to be $a(t) = A(e^{-t} - e^{-kt})$. In this equation, A is a measure of the dose of antihistamine given to the patient, and k is a transfer rate between the gastrointestinal tract and the bloodstream. A and k are positive constants, and for pharmaceuticals like antihistamine, $k > 1$.

a. [5 points] Find the location $t = T_m$ of the non-zero critical point of $a(t)$.

Solution: The maximum will occur at an endpoint or at a critical point, when $a'(t) = 0$. The critical points are thus where $a'(t) = A(-e^{-t} + ke^{-kt}) = 0$. Solving, we have $e^{-t} = ke^{-kt}$, so that $e^{(k-1)t} = k$, or $t = T_m = \frac{1}{k-1}\ln(k)$.

b. [3 points] Explain why $t = T_m$ is a global maximum of $a(t)$ by referring to the expression for $a(t)$ or $a'(t)$.

Solution: Note that $a'(0) = A(k-1) > 0$ and that for large t, $a'(t) = A(-e^{-t}+ke^{-kt}) < 0$ ($k > 1$ guarantees that the second exponential decays much faster than the first). Thus the critical point must be a maximum. In addition, because $t = T_m$ this is the only critical point we know it must be the global maximum.
Alternately, note that $a(0) = 0$. Because $k > 0$, $a(t) \geq 0$ for all t (the exponential involving $-kt$ will decay faster than e^{-t}). And for large t, $a(t) \to 0$. Thus at $t = T_m$, $a(t)$ must take on a maximum value, and because it is the only critical point this must be the global maximum.

c. [4 points] The function $a(t)$ has a single inflection point. Find the location $t = T_I$ of this inflection point. You do not need to prove that this is an inflection point.

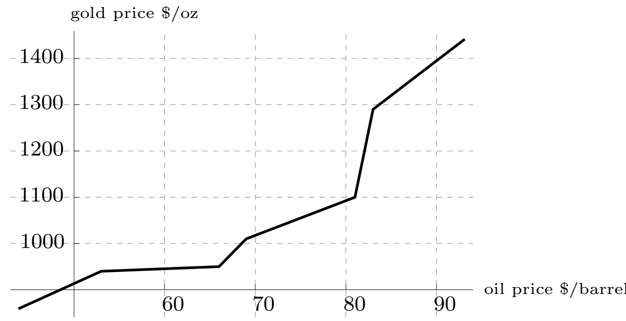
Solution: To find inflection points, we look for where $a''(t) = 0$. This gives $a''(t) = A(e^{-t} - k^2e^{-kt}) = 0$. Solving for t, we have (similarly to in (a)) $e^{(k-1)t} = k^2$, so that $t = T_I = \frac{1}{k-1}\ln(k^2) = \frac{2}{k-1}\ln(k)$.
(We could show that this is an inflection point by a similar argument to (b): because $k > 1$ we know that $a''(0) < 0$ and as t increases $a''(t)$ must eventually become positive. Thus this is an inflection point.)

d. [3 points] Using your expression for T_m from (a), find the rate at which T_m changes as k changes.

Solution: We have $T_m = \frac{1}{k-1}\ln(k)$. Thus

$$\frac{dT_m}{dk} = -\frac{1}{(k-1)^2}\ln(k) + \frac{1}{k(k-1)}.$$

5. [15 points] The graph to the right shows a function $G(b)$ that approximates the price of an ounce of gold (in dollars) as a function of the cost of a barrel of oil for data between 2009 and 2011.[1]



a. [3 points] Estimate $G'(70)$.

Solution: From the graph, it appears that between $b = 70$ and $b = 80$, G increases by about 70 as b increases about 10. Thus we estimate that $G'(70) \approx 7$ $/oz per $/barrel.

b. [5 points] Recall that G^{-1} is defined to be a function such that $G^{-1}(G(b)) = b$ (or such that $G(G^{-1}(y)) = y$, where y is the price of an ounce of gold). Derive, using the chain rule, a formula for $(G^{-1})'$ in terms of G'.

Solution: We know that $G^{-1}(G(b)) = b$. Thus $\frac{d}{db}G^{-1}(G(b)) = 1$. Differentiating the left-hand side of this using the chain rule, we have $\frac{d}{db}G^{-1}(G(b)) = (G^{-1})'(G(b)) \cdot G'(b) = 1$. Thus $(G^{-1})'(G(b)) = 1/G'(b)$.
Alternately, if we start with $G(G^{-1}(y)) = y$, we have $\frac{d}{dy}G(G^{-1}(y)) = 1$. Applying the chain rule to the left-hand side, we have $G'(G^{-1}(y)) \cdot (G^{-1})'(y) = 1$, so that $(G^{-1})'(y) = 1/G'(G^{-1}(y))$. (Obviously, with $y = G(b)$, this is the same as the previous expression.)

c. [4 points] Using parts (a) and (b), estimate $(G^{-1})'(G(70))$.

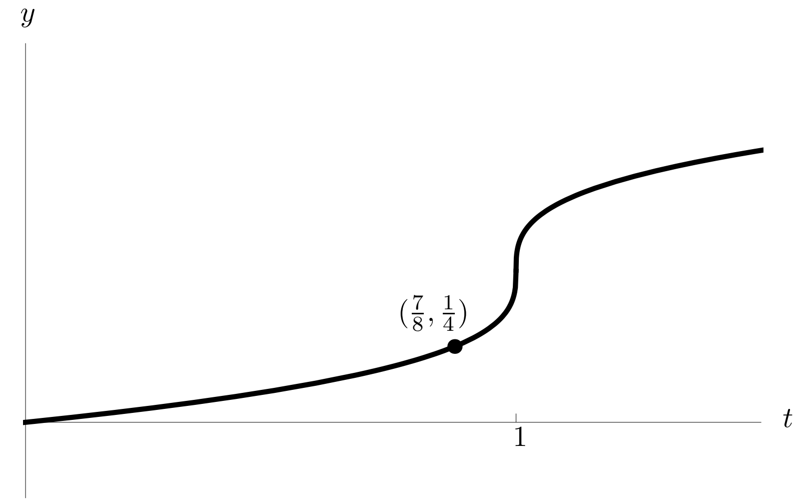
Solution: Using part (b), we have $(G^{-1})'(G(70)) = 1/G'(70) = 1/7$ $/barrel per $/oz.

d. [3 points] Explain the practical meaning of your result in (c).

Solution: $(G^{-1})'(G(70)) = 0.14$ indicates that when the price of oil is 70 $/barrel, the price of a barrel of oil goes up by about $0.14 if the price of gold goes up by $1.

[1] Gold prices from <http://www.goldprice.org/>; oil from <http://en.wikipedia.org/wiki/Price_of_petroleum>.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

Solution: The local linearization is the tangent line to the curve. We know this line has slope $h'(\frac{7}{8}) = \frac{2}{3}$ and it goes through the point $(\frac{7}{8}, \frac{1}{4})$, so it has equation

$$y - \frac{1}{4} = \frac{2}{3}(t - \frac{7}{8})$$

using point slope form. Solving for y we have $y = \frac{2}{3}t - \frac{1}{3}$. So $j(t) = \frac{2}{3}t - \frac{1}{3}$. stuff

b. [4 points] Use your answer from (a) to approximate $h(1)$.

Solution: Since $j(t)$ approximates $h(t)$ for t-values near $\frac{7}{8}$, we have

$$h(1) \approx j(1) = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}.$$

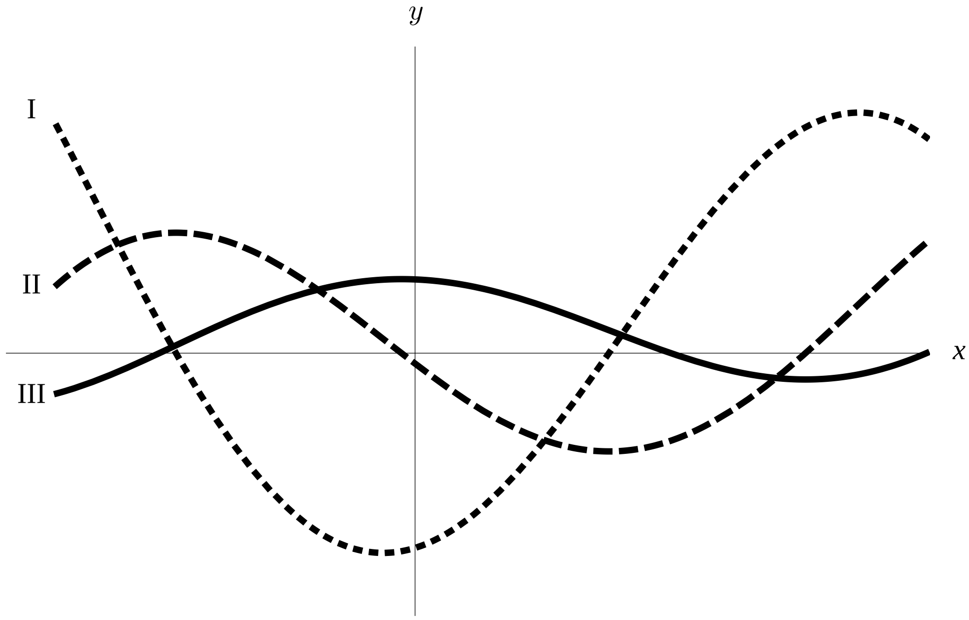
c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

Solution: The approximation in (b) is an underestimate. The function $h(t)$ is concave up at $t = 7/8$ which means the graph lies above the local linearization for t-values near $7/8$. Since we are using the local linearization to estimate the function value, our estimate will be less than the actual function value.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution: The estimate at $t = 3/4$ will be more accurate. This can be seen by drawing the tangent line and measuring the vertical distance between the estimated value and the function value at the t values $3/4$ and 1. The line is much closer to the function at $t = 3/4$ than it is at $t = 1$.

7. [12 points] On the axes below are graphed f, f', and f''. Determine which is which, and justify your response with a brief explanation.



Solution: Looking to the far right of the graph, curve **I** has a critical point where it has a slope of zero. At this x-coordinate neither of the other graphs has a root. This means the derivative of **I** is not in this figure, so **I** must be f''. Looking to the far left of the graph, **II** has a local maximum where its derivative is zero. Although **III** has a root near the same x-value, **III** changes sign from negative to positive at this point. By the first derivative test, **III** cannot be the derivative of **II**. Thus, by process of elimination, **II** must be f' and **III** must be f.

f: **III**

f': **II**

f'': **I**