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1. [12 points] Cyanide is used in solution to isolate elemental gold in gold mines. This unfortunately may result the groundwater near mines being contaminated with cyanide, which then must be removed. Suppose that at a certain mine site cyanide is removed from the groundwater starting in 2005. The concentration, c (in ppm), of cyanide found in the groundwater at the site t years after the year 2005 is given in the following table.

t (years)	0	1	2
c (ppm)	25.0	21.8	18.9

(Values for c are rounded.)

a. [4 points] Find the equation of an exponential model for $c(t)$. Show your work.

Solution: Let $c(t) = c_0\, a^t$. Then $c(0) = c_0 = 25$ and $c(1) = 25\,a = 21.8$, so $a = 21.8/25 = 0.87$. Thus $c(t) = 25\,(0.87)^t$. Note that $c(2) = 25\,(0.87)^2 = 18.9$, the value given in the table. Alternately, let $c(t) = c_0\, e^{kt} = 25\, e^{kt}$. Then $c(1) = 25\, e^k = 21.8$, so that $k = \ln(21.8/25) = \ln(0.87) = -0.14$. Thus we may also write $c(t) = 25\, e^{-0.14t}$. As before, this gives $c(2) = 18.9$.

b. [4 points] Using your equation from (a), how many years will it take for the concentration of cyanide to be reduced to 10 ppm? Show all of your work.

Solution: We want $c(t) = 25\,(0.87)^t = 10$, so that $t = \ln(10/25)/\ln(.87) = 6.6$ years. Alternately, with $c(t) = 25\, e^{-0.14t} = 10$, we have $t = \ln(10/25)/(-0.14) = 6.6$ years as well.

c. [2 points] The cyanide removal process involves pumping groundwater through a filtering system. Suppose that the speed of this pumping process is doubled from the pumping speed which produced the data given above. Call the resulting concentration function $c_1(t)$. Use your expression for $c(t)$ from (a) to write an equation for $c_1(t)$.

Solution: If we double the pumping rate, then we expect our new $c(t)$ to be $c_1(t) = c(2t) = 25\,(0.87)^{2t}$ $(= 25(0.76)^t = 25e^{-0.27t})$.

d. [2 points] Now instead suppose that the groundwater cleaning started 3 years earlier. Call the resulting concentration function $c_2(t)$. Use your expression for $c(t)$ from (a) to write an equation for $c_2(t)$.

Solution: Starting 3 years earlier, we have $c_2(t) = c(t+3) = 25\,(0.87)^{t+3}$ $(= 16.5\,(0.87)^t)$.

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2. [12 points] Consider the following table giving values, rounded to three decimal places, of a function $f(x)$.

x	0	0.5	1
$f(x)$	0	0.247	0.841

a. [3 points] Estimate $f'(1)$. Be sure it is clear how you obtain your answer.

Solution: We can estimate the derivative $f'(1)$ with a difference quotient taking $h = -0.5$: $f'(1) \approx \frac{0.841-0.247}{1-0.5} = 1.188$.

b. [4 points] Estimate $f''(1)$. Again, be sure that it is clear how you obtain your answer.

Solution: We procede similarly to part (a), but need the value of $f'(0.5)$ to complete the difference quotient. Using a central difference for $f'(0.5)$ we have $f'(0.5) \approx \frac{0.841-1}{1} = 0.841$, so that

$$f''(1) \approx \frac{f'(1) - f'(0.5)}{1 - 0.5} = \frac{1.188 - 0.841}{0.5} = 0.694.$$

Alternately, if we use $h = -0.5$ for $f'(0.5)$ as well as for $f'(1)$, we have $f'(0.5) \approx \frac{.247-0}{0.5-0} = 0.494$, so that

$$f''(1) \approx \frac{f'(1) - f'(0.5)}{1 - 0.5} = \frac{1.188 - 0.494}{1 - 0.5} = 1.388.$$

c. [3 points] Estimate $f(1.25)$, being sure your work is clear.

Solution: We know that $f(1) = 0.841$ and have $f'(1) \approx 1.188$. We can therefore estimate that

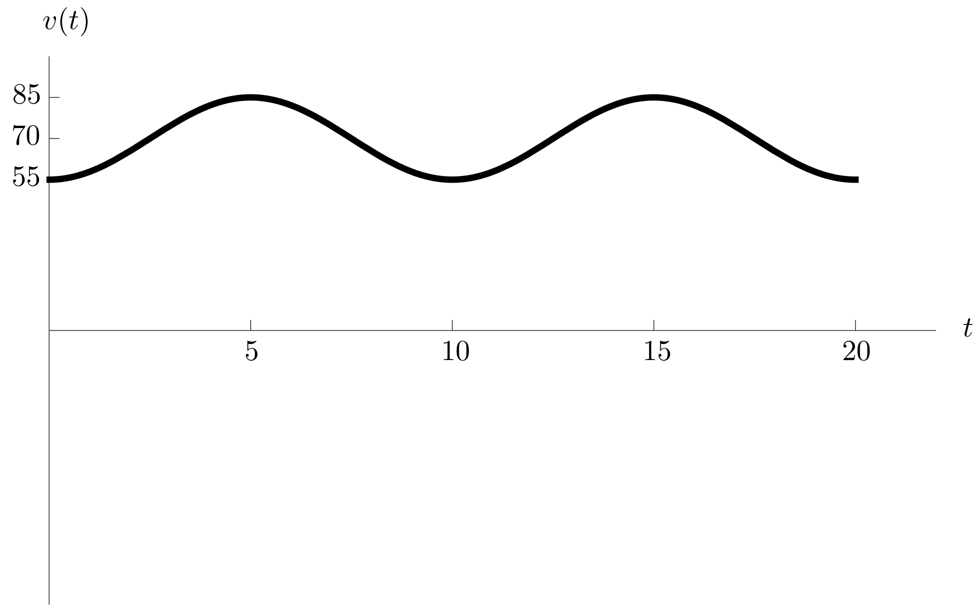
$$f(1.25) \approx 0.841 + (0.25)(1.188) = 1.138.$$

d. [2 points] Based on your work in (a) and (b), is your estimate in (c) an over- or under-estimate? Explain.

Solution: Because $f''(1) \approx 1.388$, we expect that the actual slope between $x = 1$ and $x = 1.25$ is more than our value of 1.388, and thus that this is an underestimate for the actual value of $f(1.25)$.

3. [12 points] Suppose that when you merge onto the highway the blue car in front of you is moving at 55 mph. Immediately after you merge, the driver of the blue car speeds up until, after five minutes, it is going 85 mph. Then, during the next five minutes it slows down to 55 mph again. This process then repeats over the following 10 minutes, with the blue car speeding up to 85 mph and then decreasing to 55 mph again.

a. [6 points] Assuming the speed of the blue car follows a sinusoidal pattern, on the axes below draw a well-labeled sketch of two periods of a function $v(t)$ which outputs the speed of the car t minutes after you merge onto the highway.



b. [6 points] Find a possible formula for $v(t)$ from part (a). What are the period and amplitude of $v(t)$?

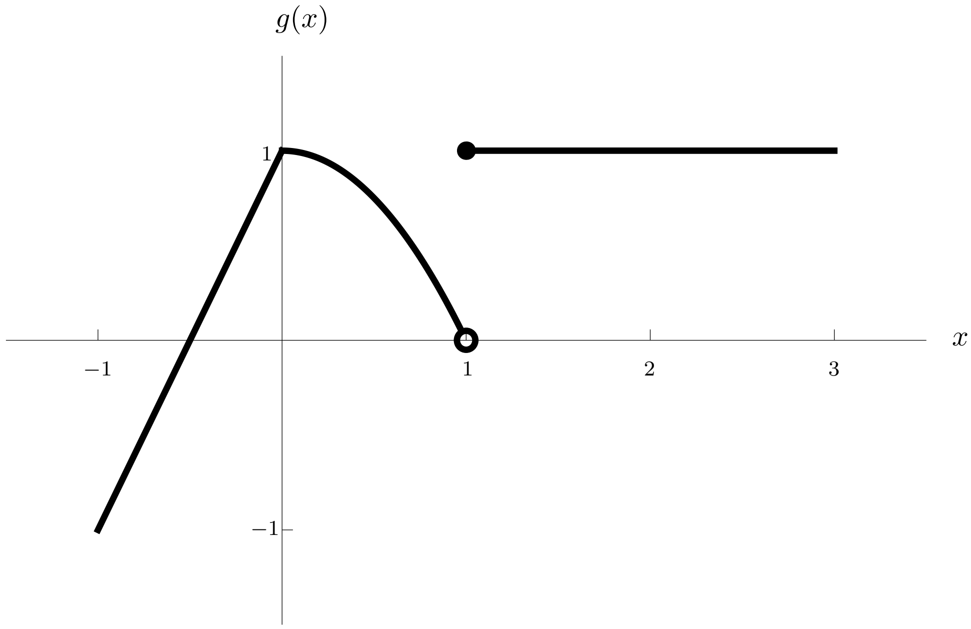
$v(t) =$ $-15\cos(\frac{\pi}{5}t) + 70$

period = **10 min**

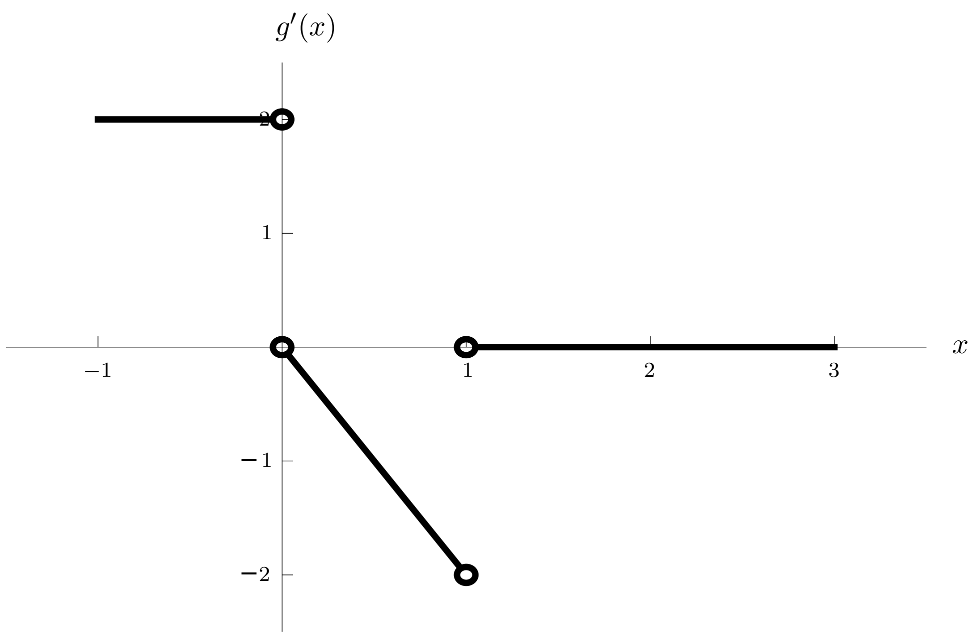
amplitude = **15 mph**

Solution: Notice that the oscillation is between 55 and 85 mph, so that the midline value is 70 mph. Further, the oscillation starts at the minimum value, so that the easiest formula for the velocity will be $v(t) = -A\cos(b\,t) + 70$. Here A is the amplitude of the oscillation, which is $(85 - 55)/2 = 15$. Finally, $b = 2\pi/T$, where T is the period, which is given to be 10 minutes. Thus $b = \pi/5$. The period is 10 min, and the amplitude is 15 mph. Because of the periodicity of sine and cosine, equivalent formulae would be $v(t) = 15\sin(\frac{\pi}{5}\,(t - 2.5)) + 70$ and $v(t) = 15\cos(\frac{\pi}{5}\,(t - 5)) + 70$.

4. [10 points] The graph of a function $g(x)$ is given below.



Accurately sketch a graph of $g'(x)$ on the axes below. Be sure to label the vertical axis.



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5. [12 points] A paperback book (definitely not a valuable calculus textbook, of course) is dropped from the top of Dennison hall (which is 40 m high) towards a very large, upward pointing fan. The average velocity of the book between time $t = 0$ and later times is shown in the table of data below (in which t is in seconds and the velocities are in m/s).

between $t = 0$ seconds and $t =$	1	2	3	4	5
average velocity is	-5	-10	-11.67	-9	-7.2

a. [8 points] Fill in the following table of values for the height $h(t)$ of the book (measured in meters). Show how you obtain your values.

t	0	1	2	3	4	5
$h(t)$	40	**35**	**20**	**5**	**4**	**4**

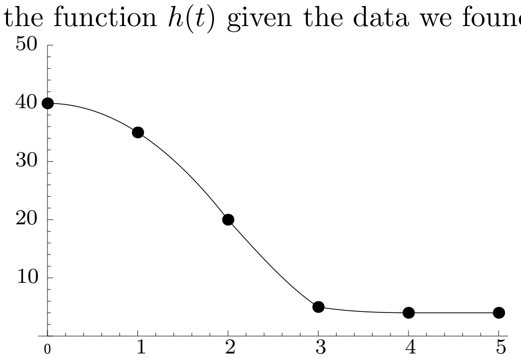
Solution: For each value, we use the definition of average velocity:

$$\text{average velocity on } [0, a] = \frac{h(a) - h(0)}{a}.$$

Thus, the average velocity between $t = 0$ and $t = 1$ gives us $h(1) - 40 = -5$, so $h(1) = 35$. Similarly, between $t = 0$ and $t = 2$ we have $(h(2) - 40)/2 = -10$, so that $h(2) = 20$, etc.

b. [4 points] Based on your work from (a), is $h''(1) > 0$, < 0, or $= 0$? Is $h''(3) > 0$, < 0, or $= 0$? Explain.

Solution: A sketch of the function $h(t)$ given the data we found in (a) is shown below.



We see that $h(t)$ is concave down at $t = 1$ and concave up at $t = 3$. Thus $h''(1) < 0$ and $h''(3) > 0$.

Alternate solution: The average velocity between $t = 0$ and $t = 1$ is -5 and approximates $h'(0.5)$. The average velocity between $t = 1$ and $t = 2$ is $(20-15)/(2-1) = -15 \approx h'(1.5)$. Thus the velocity appears to be decreasing at $t = 1$, so that $h''(1) < 0$. Similarly we have $h'(2.5) \approx -15$ and $h'(3.5) \approx -1$, so $h''(3) > 0$.

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6. [12 points] For the graph $y = f(x)$ in the figure below, arrange the following numbers from smallest to largest:

A. The slope of the graph at A.

B. The slope of the graph at B.

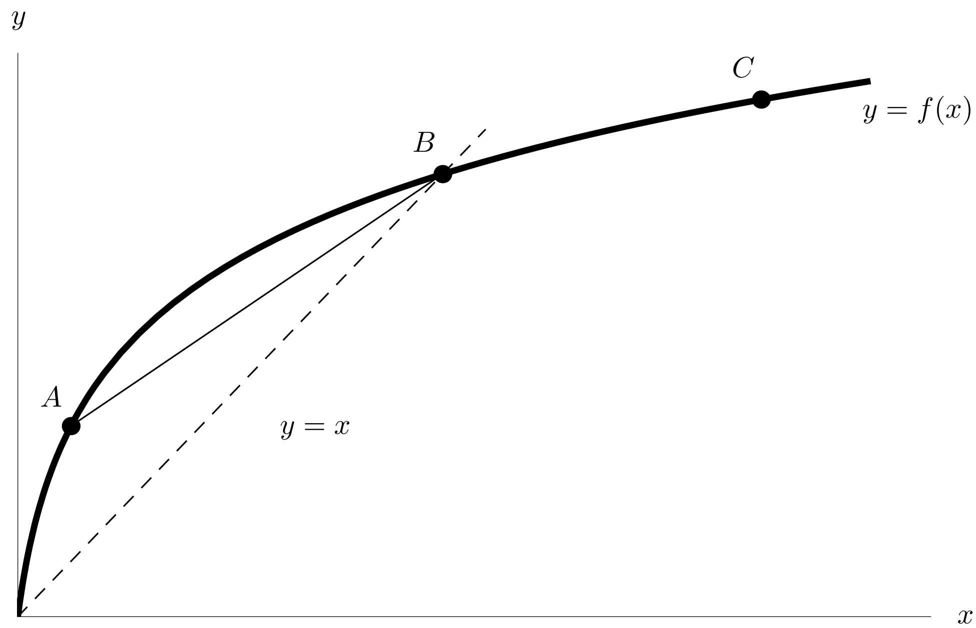
C. The slope of the graph at C.

AB. The slope of the line AB.

0. The number 0.

1. The number 1.

Explain the positions of the number 0 and the number 1 in your ordering. Any unclear answers will be counted as incorrect.



0 < **C** < **B** < **AB** < **1** < **A**

Solution: The number one and all other slopes are positive, so 0 must be the smallest number. The line $y = x$ has a slope of 1. The slope at C, the slope at B, and the slope of the line AB are each smaller than the slope of the line $y = x$ by looking at the picture. The slope at A is larger than the slope of $y = x$ also by the picture. Thus 1 is the second to largest number in the ordering.

7. [12 points] For each of the descriptions of a function f that follow, indicate which of the graphs below match the description. For each description there may be no, one, or several graphs that match; write **none** if no graphs match the description. You may need to use a graph more than once. In each case you should assume that f is defined only on the domain $[0, 2]$.

- $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$; $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$; and $f(x)$ is continuous everywhere except at $x = 1$.

  matching graph(s): **A**

- $f''(x) > 0$ for all $x \neq 1$; $f(x) < 0$ for all $x \neq 1$; and $f(x)$ is differentiable everywhere except at $x = 1$.

  matching graph(s): **C,E**

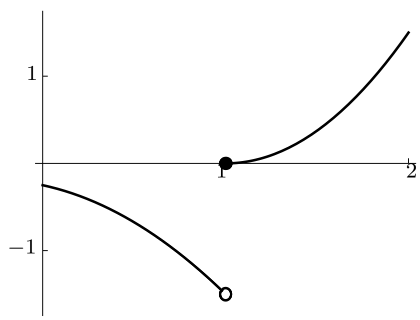
- $f''(x) < 0$ for all $x \neq 1$; $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$; and $f(x) < 0$ for all $x \neq 1$.

  matching graph(s): **none**

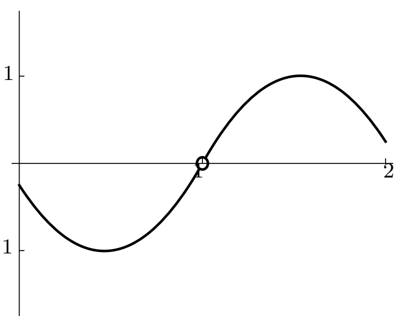
- $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$; $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$; and $f(x)$ is differentiable everywhere except at $x = 1$.

  matching graph(s): **A,D**

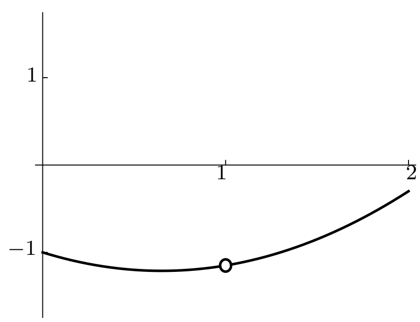
A.



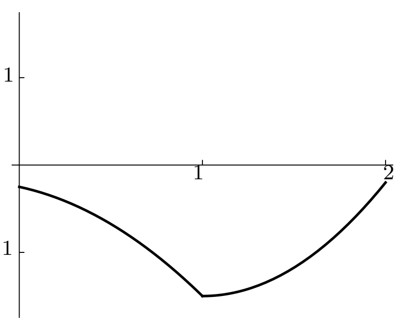
B.



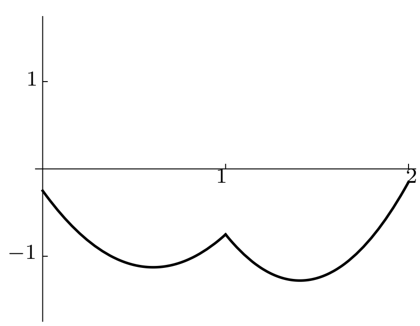
C.



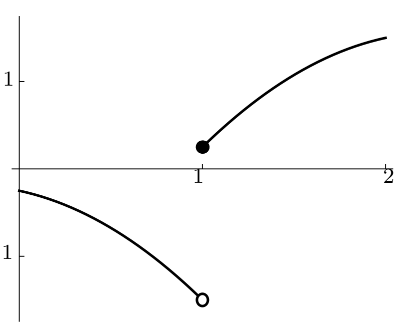
D.



E.



F.



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8. [12 points] Let $P(d)$ be a function giving the total electricity that a solar array has generated, in kWH, between the start of the year and the end of the dth day of the year. Each of the following centences (a)–(d) expresses a mathematical equality in practical terms. For each, give a **single** mathematical equality involving P (and, as needed, its inverse and derivatives) that corresponds to the sentence.

a. [3 points] The end of the day on which the array had generated 3500 kWH of electricity was the end of the 4th of January.

Solution: $P^{-1}(3500) = 4$. ($P(4) = 3500$ is equivalent, though the statement suggests that 3500 kWH is the independent variable, and so that the equation involving the inverse is the better for this statement.)

b. [3 points] At the end of January 4th, the array was generating electricity at a rate of 1000 kWH per day.

Solution: $P'(4) = 1000$.

c. [3 points] When the array had generated 5000 kWH of electricity, it would take approximately half a day to generate an additional 1000 kWH of electricity.

Solution: $(P^{-1})'(5000) = \frac{1}{2000}$. (Or, alternately, $P'(P^{-1}(5000)) = 2000$.)

d. [3 points] At the end of January 30th, it would take approximately one day to generate an additional 2500 kWH of electricity.

Solution: $P'(30) = 2500$. (Or, $P'(31) = 2500$.)

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9. [6 points] The population, $P(t)$, of China, in billions, can be approximated by

$$P(t) = 1.267(1.007)^t,$$

where t is the number of years since the start of 2000.

a. [2 points] Calculate the *continuous growth rate* of $P(t)$.

Solution: The function $P(t)$ is an exponential function of the form $P_0 a^t$ (which can be written as $P_0\, e^{kt}$). The continuous growth rate of such an exponential is $\ln a$ $(= k)$. In this case since $a = 1.007$, the continuous growth rate is $\ln 1.007$.

b. [4 points] Using the limit definition of the derivative, write an explicit expression for the derivative of $P(t)$ at the beginning of 2011. You do not need to simplify your expression.

Solution: The beginning of 2011 corresponds to $t = 11$ in this problem. The definition of the derivative of P at $t = 11$ is

$$\begin{aligned} P'(11) &= \lim_{h \to 0} \frac{P(11+h) - P(11)}{h}. \\ &= \lim_{h \to 0} \frac{1.267(1.007)^{11+h} - 1.267(1.007)^{11}}{h}. \end{aligned}$$