1. (24 pts - 6pts ea) Assume y is a function of x, find y' given:

(a) $y = \dfrac{1}{\ln(x)}$ (b) $xe^{y^2} = \tan^{-1}(e^x) - y$ (c) $y = (\cos x)^x$ (d) $y = \displaystyle\int_e^{e^x} (t)^{\ln(t)}\, dt$

Solution:

(a) Using the quotient rule, $y' = \dfrac{0-(1/x)}{(\ln(x))^2} = \boxed{-\dfrac{1}{x\ln^2(x)}.}$

(b) Using implicit differentiation we have $e^{y^2} + xe^{y^2}2yy' = \dfrac{e^x}{1+e^{2x}} - y'$ so,

$$(xe^{y^2}2y+1)y' = \frac{e^x}{1+e^{2x}} - e^{y^2} \implies \boxed{y' = \frac{1}{2xye^{y^2}+1}\left(\frac{e^x}{1+e^{2x}} - e^{y^2}\right)}$$

(c) Using logarithmic differentiation we have, $\ln(y) = x\ln(\cos x)$ and so $\dfrac{y'}{y} = \ln(\cos x) + x\dfrac{-\sin(x)}{\cos(x)}$ and thus, $\boxed{y' = (\cos x)^x\,(\ln(\cos x) - x\tan x)\,.}$

(d) Using the Fundamental Theorem of Calculus we have, $y' = (e^x)^{\ln(e^x)}e^x = (e^x)^x e^x = \boxed{e^{x^2+x}.}$

2. (24 pts - 6pts ea) Evaluate the integrals:

(a) $\displaystyle\int \frac{\ln(x^2e^{\sqrt{x}})}{x}\,dx$ (b) $\displaystyle\int \frac{dx}{x^{3/2}+x^{1/2}}$ (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$ (d) $\displaystyle\int_0^1 \ln(\sinh(x)+\cosh(x))\,dx$

Solution:

(a) Using the properties of the log we have,

$$\int \frac{\ln(x^2e^{\sqrt{x}})}{x}\,dx = 2\int\frac{\ln x}{x}\,dx + \int\frac{\sqrt{x}}{x}\,dx \underbrace{=}_{u=\ln x} 2\int u\,du + \int x^{-1/2}\,dx = 2\frac{u^2}{2} + 2x^{1/2} + C = \boxed{\ln^2(x) + 2x^{1/2} + C}$$

(b) $\displaystyle\int \frac{dx}{x^{3/2}+x^{1/2}} = \int\frac{dx}{x^{1/2}(x+1)} \underbrace{=}_{u=x^{1/2}} = \int\frac{2du}{u^2+1} = 2\tan^{-1}(u) + C = \boxed{2\tan^{-1}(\sqrt{x}) + C}$

(c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx \underbrace{=}_{u=2x+1} \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\,du = \frac{1}{4}\int_1^9 (\sqrt{u} - u^{-1/2})\,du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]\Bigg|_1^9 = \boxed{\frac{10}{3}}$

(d) $\displaystyle\int_0^1 \ln(\sinh(x)+\cosh(x))\,dx = \int_0^1 \ln(e^x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Bigg|_0^1 = \boxed{\frac{1}{2}}$

3. (24 pts - 6pts ea) Find the limits

(a) $\displaystyle\lim_{x\to\infty}(1-2x)^{1/x}$ (b) $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt{1+x^2}}$ (c) $\displaystyle\lim_{x\to\infty}\frac{1}{2\tan^{-1}(x)-\pi}$ (d) $\displaystyle\lim_{x\to e}\frac{1}{x-e}$

Solution:

(a) TYPO: Should be $\displaystyle\lim_{x\to\infty}(1+2x)^{1/x}$. Now, let $y = \displaystyle\lim_{x\to\infty}(1+2x)^{1/x}$, then

$$\ln(y) = \lim_{x\to\infty}\frac{\ln(1+2x)}{x} \stackrel{L'H}{=} \lim_{x\to\infty}\frac{2/(1+2x)}{1} = 0$$

so $y = e^0 = 1$.

(b) Note that L'Hospitals Rule will not work here, so

$$\lim_{x\to-\infty}\frac{x}{\sqrt{1+x^2}} = \lim_{x\to-\infty}\frac{x}{\sqrt{x^2}\sqrt{1/x^2+1}} \underbrace{=}_{\sqrt{x^2}=|x|} \lim_{x\to-\infty}\frac{x}{|x|\sqrt{1/x^2}} \underbrace{=}_{|x|=-x \text{ if } x<0} \lim_{x\to-\infty}\frac{x}{-x\sqrt{1/x^2+1}} = -1$$

(c) $\displaystyle\lim_{x\to\infty}\frac{1}{2\tan^{-1}(x)-\pi} = -\infty$ since $2\tan^{-1}(x) < \pi$ for all x.

(d) We need to check the one sided limits,

$$\lim_{x\to e^-}\frac{1}{x-e} = -\infty \text{ and } \lim_{x\to e^+}\frac{1}{x-e} = +\infty$$

so the limit does not exist.

4. (a) (5 pts) Find the linearization of $f(x) = \ln(1-x)$ at $a = 0$.

(b) (5 pts) Use your linearization from part (a) to approximate $\ln(0.99)$.

Solution:

(a) The linearization is $L(x) = f(0) + f'(0)x = 0 + (-1)\cdot x = \boxed{-x.}$

(b) From (a) we have, $\ln(0.99) = f(0.01) \approx L(0.01) = \boxed{-0.01.}$

5. (a) (5 pts) Consider a cylinder with a circular base and a fixed height of 7 inches. Suppose the radius of the base is r. If V is the volume of the cylinder, find the rate of change of volume in terms of the rate of change of the radius r with respect to time. (Note, $V = 7\pi r^2$)

(b) (10 pts) Suppose the radius of the cylinder mentioned in part (a) is measured to be 0.8 inches with an error in measurement of 0.01 inches, use differentials to estimate the percentage error in calculating the volume of the cylinder.

Solution:

(a) Here $V = 7\pi r^2$ and so $\boxed{\dfrac{dV}{dt} = 14\pi r\dfrac{dr}{dt}.}$

(b) Here $r = 0.8$ and $dr = \Delta r = 0.01$ and $dV = 14\pi r dr$ and so

$$\frac{dV}{V} = \frac{14\pi r dr}{7\pi r^2} = 2\frac{dr}{r} = 2\frac{0.01}{0.8} = \frac{1}{40} = 0.025$$

so the percentage error in calculating the volume of the cylinder is $\boxed{2.5\%.}$

6. The velocity function of a particle moving along a straight line is given by $v(t) = t^2 + 3t - 4$ m/s with initial position $s(0) = 106$ m.

(a) (5 pts) Find the position of the particle at any time t

(b) (5 pts) Find the acceleration of the particle at any time t

(c) (10 pts) Find the total distance travelled by the particle during the first 3 seconds.

Solution:

(a) $s(t) = \displaystyle\int v(t) = \frac{t^3}{3} + \frac{3t^2}{2} - 4t + C$ and so $\boxed{s(t) = \dfrac{t^3}{3} + \dfrac{3t^2}{2} - 4t + 106.}$

(b) $a(t) = v'(t) = \boxed{2t+3.}$

(c) $\displaystyle\int_0^3 |v(t)|dt = \int_0^1 -v(t)\,dt + \int_1^3 v(t)\,dt = \frac{13}{6} + \frac{38}{3} = \boxed{\frac{89}{6}.}$

7. (10 pts) Use the Mean Value Theorem to prove that if $a < b$ then $\sin(b) - \sin(a) \le b - a$.

Solution:

If we let $f(x) = \sin(x)$, then the Mean Value Theorem applied to $f(x)$ on the interval $[a, b]$ implies,

$$\frac{\sin(b) - \sin(a)}{b - a} = \frac{f(b) - f(a)}{b - a} = f'(c) = \cos(c) \le 1$$

so $\dfrac{\sin(b) - \sin(a)}{b - a} \le 1$ which implies $\sin(b) - \sin(a) \le b - a$.

8. Given that $f(x) = \dfrac{x^2}{2} + \dfrac{\ln[(x-2)^2]}{2}$, $f'(x) = x + \dfrac{1}{(x-2)}$, and $f''(x) = 1 - \dfrac{1}{(x-2)^2}$

(a) (2 pts) State the domain of $f(x)$.

(b) (4 pts) Does $f(x)$ have any *vertical asymptotes*? Justify your answer with a limit.

(c) (4 pts) Does $f(x)$ have any *horizontal asymptotes*? Justify your answer with a limit.

(d) (4 pts) On what interval (or intervals) is $f(x)$ *increasing* and/or *decreasing*? Justify your answer.

(e) (4 pts) On what interval (or intervals) is $f(x)$ *concave up* and/or *concave down*? Justify your answer.

(f) (2 pts) Find all local maximum or minimum values of $f(x)$.

(g) (3 pts) Sketch the graph of $f(x) = \dfrac{x^2}{2} + \dfrac{\ln[(x-2)^2]}{2}$. (Clearly label and sketch your graph.)

Solution:

(a) $x \ne 2$

(b) Yes, $x = 2$ since $\lim_{x \to 2} f(x) = -\infty$.

(c) No, since $\lim_{x \to \pm\infty} f(x) = +\infty$.

(d) Decreasing on $(-\infty, 2)$ and increasing on $(2, +\infty)$.

(e) Concave up on $(-\infty, 1) \cup (3, +\infty)$ and concave down on $(1, 3)$

(f) No local extrema.

(g) The graph,