1. (15 points) Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-2, 1]$.

$$f'(x) = 3x^2 - 6x$$

$$0 = f'(x) = 3x(x-2)$$

$$x = 0, \underbrace{2}_{\text{not in } [-2,1]}$$

Plug in crit pts and endpoints

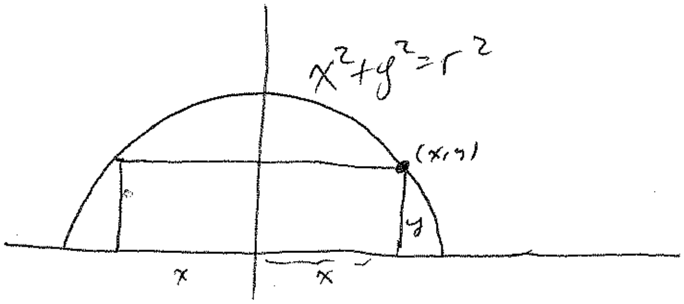
$$f(-2) = (-2)^3 - 3(-2)^2 + 1 = -19 \quad \leftarrow \text{min}$$

$$f(0) = 1 \quad \leftarrow \text{max}$$

$$f(1) = -1$$

$$\boxed{\begin{array}{l}\text{Absolute Min at } (-2, -19) \\ \text{Abs Max at } (0, 1)\end{array}}$$

2. (20 points) Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



$$A = 2xy \qquad x^2+y^2=r^2$$

$$y = \sqrt{r^2-x^2}$$

$$\underline{A(x) = 2x\sqrt{r^2-x^2}} \quad \leftarrow \text{Maximize this}$$

$$A' = 2\sqrt{r^2-x^2} + 2x\,\tfrac{1}{2}(r^2-x^2)^{-1/2}(-2x)$$

$$0 = 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}}$$

$$\frac{x^2}{\sqrt{r^2-x^2}} = \sqrt{r^2-x^2}$$

$$x^2 = r^2-x^2$$

$$x^2 = \frac{r^2}{2}$$

$$\underline{x = \frac{r}{\sqrt{2}}} \quad \Rightarrow y = \sqrt{r^2-\left(\frac{r}{\sqrt{2}}\right)^2} = \sqrt{r^2-\frac{r^2}{2}} = \frac{r}{\sqrt{2}}$$

$$\Rightarrow \boxed{A\left(\frac{r}{\sqrt{2}}\right) = 2\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{\sqrt{2}}\right) = r^2}$$

3. (5 points) Using the fact that $\frac{d}{dx}((\sin(x)) = \cos(x)$, show that there exists some c in the closed interval $[0, \frac{\pi}{2}]$ such that $\cos(c) = \frac{2}{\pi}$.

Let $f(x) = \sin x$, which is diff'le.

Mean Value Theorem implies exists c in $[0, \pi/2]$ st

$$\frac{f(\pi/2) - f(0)}{\pi/2 - 0} = f'(c)$$

$$\frac{\sin(\pi/2) - \sin(0)}{\pi/2} = \cos(c)$$

$$\frac{1}{\pi/2} = \boxed{\frac{2}{\pi} = \cos(c)}$$

4. (15 points) Consider the function $f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2$.
(a) Find where $f(x)$ is increasing and decreasing and the x-values of any local extrema.
(b) Find where $f(x)$ is concave up and concave down and the x-values of any inflection points.
(c) Sketch a graph of $f(x)$. Make sure your graph reflects the information obtained above.

a. $f' = x^3 - 6x^2 + 9x$

$0 = f' = x(x^2 - 6x + 9)$

$0 = x(x-3)^2$

$\underline{x = 0, 3}$

f ↘ min ↗ ↗

$-\infty$ 0 3 $+\infty$

f' $-$ $+$ $+$

$$\boxed{\begin{array}{ll} \text{Inc} & (0,3),\ (3,+\infty) \\ \text{Dec} & (-\infty, 0) \\ \text{Local min} & x=0 \end{array}}$$

b. $f'' = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$0 = 3(x-1)(x-3)$

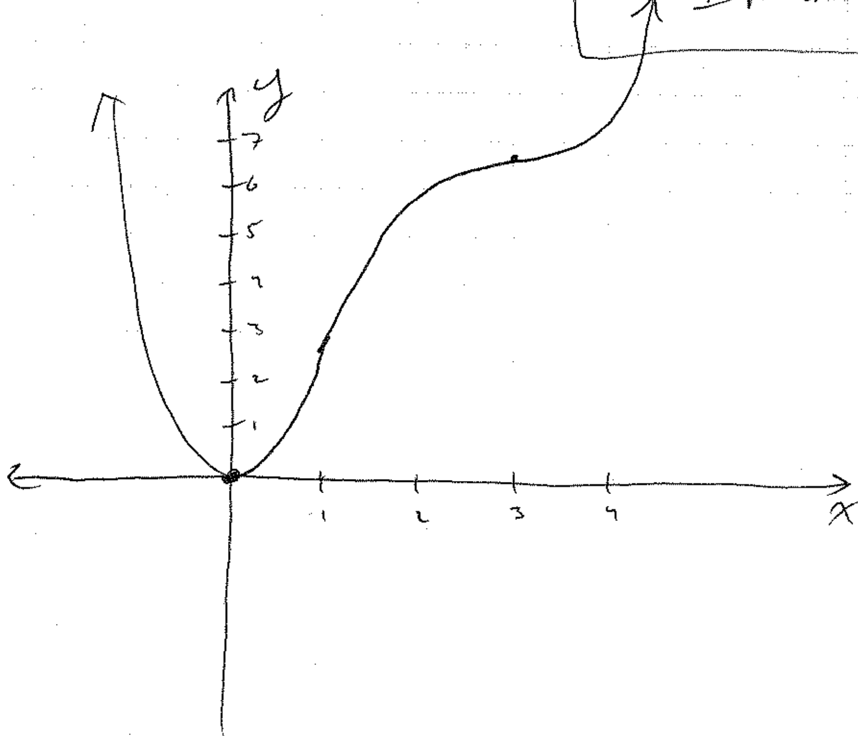
$\underline{x = 1, 3}$

f ∪ IP ∩ IP ∪

$-\infty$ 1 3 $+\infty$

f'' $+$ $-$ $+$

$$\boxed{\begin{array}{ll} f \text{ Conc Up} & (-\infty, 1),\ (3, +\infty) \\ \text{Conc Down} & (1,3) \\ \text{IP at} & x = 1, 3 \end{array}}$$



5. (15 points) Compute the following limits:

(a) $\lim_{x \to 1} \frac{x^3 - 3x + 2}{x - 1} =$ (Plug-in $= \frac{1^3 - 3(1) + 2}{1 - 1} = \frac{0}{0}$)

L'Hopital's $\stackrel{}{=}$ $\lim_{x \to 1} \frac{3x^2 - 3}{1} = \frac{3(1)^2 - 3}{1} = \boxed{0}$

(b) $\lim_{x \to 0} \frac{\cos(2x) - \cos(x)}{x^2} =$ (Plug-in $= \frac{\cos 0 - \cos 0}{0^2} = \frac{0}{0}$)

L'Hopital $=$ $\lim_{x \to 0} \frac{-2\sin(2x) + \sin(x)}{2x}$ (Plug-in $= \frac{-2\sin 0 + \sin 0}{0} = \frac{0}{0}$)

L'Hopital $=$ $\lim_{x \to 0} \frac{-4\cos(2x) + \cos(x)}{2} = \frac{-4\cos 0 + \cos 0}{2}$

$= \frac{-4+1}{2} = \boxed{\frac{-3}{2}}$

6. (15 points) (a). You wish to find a solution to the equation $x^2 + x - 1 = 0$. Use Newton's method with $x_0 = 0$ and find x_1, x_2.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f = x^2 + x - 1$$
$$f' = 2x + 1$$

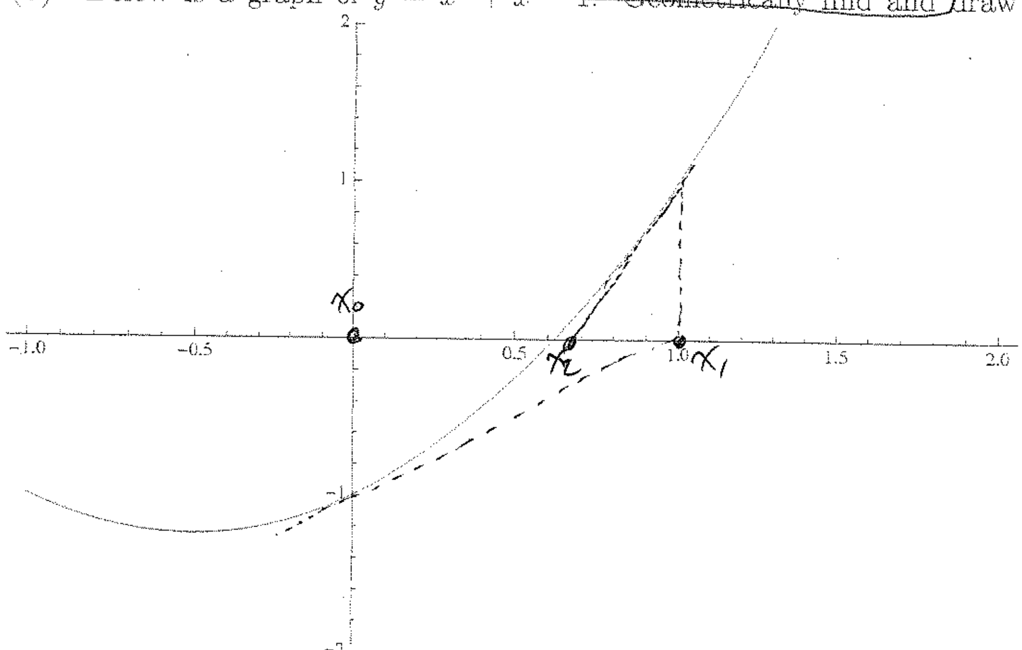
$$x_0 = 0$$

$$x_1 = 0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{1} = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{3} = 2/3$$

$$\boxed{x_0 = 0, \quad x_1 = 1, \quad x_2 = 2/3}$$

(b). Below is a graph of $y = x^2 + x - 1$. Geometrically find and draw x_0, x_1, x_2 on the graph.



7. (15 points) Find the following anti-derivatives:

(a) $\int (x^4 - 2x^2 + 3)dx$ $= \frac{1}{5}x^5 - \frac{2}{3}x^3 + 3x + C$

(b) $\int \frac{1}{\sqrt[3]{x}}dx$ $= \int x^{-1/3} dx = \frac{3}{2}x^{2/3} + C$

(c) $\int (\cos(2x) + \sin(2x))\, dx$ $= \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + C$