1. (20 points) a. If $y = x^4 - 3x^2$, then $\frac{dy}{dx} = ?$

$$\boxed{\frac{dy}{dx} = 4x^3 - 6x}$$

b. Find $\frac{dy}{dx}$ given that $x = \dfrac{y^2+1}{y-x}$.

$$x = \frac{y^2+1}{y-x}$$

$$x(y-x) = y^2+1$$

$$xy - x^2 = y^2+1$$

$$\frac{d}{dx}(xy-x^2) = \frac{d}{dx}(y^2+1)$$

$$y + x\frac{dy}{dx} - 2x = 2y\, dy/dx$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x-2y) = 2x-y$$

$$\boxed{\frac{dy}{dx} = \frac{2x-y}{x-2y}}$$

or

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\frac{y^2+1}{y-x}\right)$$

$$1 = \frac{(y-x)(2y\, dy/dx) - (y^2+1)\left(\frac{dy}{dx} - 1\right)}{(y-x)^2}$$

$$(y-x)^2 = (2y^2 - 2xy)\frac{dy}{dx} - (y^2+1)\frac{dy}{dx} + (y^2+1)$$

$$(y-x)^2 - (y^2+1) = (2y^2 - 2xy - y^2 - 1)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{(y-x)^2 - (y^2+1)}{2y^2 - 2xy - y^2 - 1}}$$

$$= \frac{\cancel{y^2} - 2xy + x^2 - \cancel{y^2} - 1}{y^2 - 2xy - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 2xy - 1}{y^2 - 2xy - 1}}$$

Note: These can be seen to be the same using substitution $y^2+1 = x(y-x)$

2. (20 points) a. Suppose $x(t) = \sin(\sin(t^3))$. Find $\frac{dx}{dt}$.

$$\frac{dx}{dt} = \cos(\sin t^3) \frac{d}{dt}(\sin t^3)$$

$$= \cos(\sin t^3) \cos t^3 \frac{d}{dt}(t^3)$$

$$\boxed{= \cos(\sin(t^3)) \cos(t^3) 3t^2}$$

b. When $g(x) = \sin(x^2)$, find $\dfrac{d^2g}{dx^2}$ at $x = \sqrt{\frac{\pi}{4}}$.

$$\frac{dg}{dx} = \cos(x^2) \cdot 2x$$

$$\frac{d^2g}{dx^2} = \frac{d}{dx}[\cos x^2] 2x + \cos x^2 \frac{d}{dx}[2x]$$

$$= -\sin(x^2) \cdot 2x \cdot 2x + \cos(x^2) \cdot 2$$

$$= -4x^2 \sin(x^2) + 2\cos(x^2)$$

$$\left.\frac{d^2g}{dx^2}\right|_{x=\sqrt{\pi/4}} = -4\left(\sqrt{\pi/4}\right)^2 \sin\left(\sqrt{\pi/4}^2\right) + 2\cos\left(\sqrt{\pi/4}^2\right)$$

$$= -4 \cdot \frac{\pi}{4} \cdot \sin(\pi/4) + 2\cos(\pi/4)$$

$$= -\pi \frac{\sqrt{2}}{2} + \cancel{2}\frac{\sqrt{2}}{\cancel{2}}$$

$$\boxed{= \frac{-\pi\sqrt{2}}{2} + \sqrt{2}}$$

3. (10 points) Find $\frac{d}{dx}(3x^2)$ using the definition of the limit.

$$\frac{d}{dx}(3x^2) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h^2}$$

$$= \lim_{h \to 0} \frac{3(x^2+2xh+h^2) - 3x^2}{h^2}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h^2}$$

$$= \lim_{h \to 0} \frac{h}{h}(6x+3h)$$

$$= \lim_{h \to 0} 6x + 3h = \boxed{6x}$$

4. (15 points) Write the equation for the tangent line to the curve $y = x^5$ at the point $x = 2$.

$f(x) = x^5$ $\quad$ $f(2) = 2^5 = 32$

$f'(x) = 5x^4$ $\quad$ $f'(2) = 5 \cdot 2^4 = 80$

Tangent line goes through point $(2, f(2)) = (2, 32)$
w/ slope 80

$\boxed{y - 32 = 80(x-2)}$

$\boxed{y = 80(x-2) + 32}$

b. Use linear approximation to estimate $(2.001)^5$.

$L(x) = 80(x-2) + 32$

$(2.001)^5 = f(2.001) \approx L(2.001) = 80(2.001-2) + 32$

$= 80(.001) + 32$

$= .08 + 32$

$= 32.08$

$\boxed{(2.001)^5 \approx 32.08}$

5. (15 points) If a ball is thrown vertically upward with an initial velocity of 64 ft/sec, then its height in feet after t seconds is

$$s(t) = 64t - 16t^2.$$

What is the maximum height reached by the ball?

Max ht occurs when $v(t) = 0$

$$v(t) = s'(t) = 64 - 32t$$

$$0 = v(t) = 64 - 32t$$

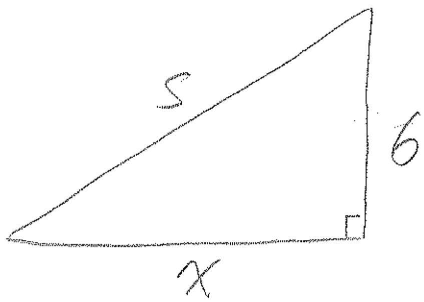
$$t = \frac{64}{32} = 2$$

Time when max height occurs

$$s(2) = 64(2) - 16(2^2) = 2 \cdot 64 - 64 = 64$$

$\boxed{\text{Max ht is } s(2) = 64 \text{ ft}}$

6. (20 points) A boat is pulled toward a dock by a rope from the bow (tip of the boat) through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec. What is the boat's speed when 10 ft of rope are out?



$\frac{ds}{dt} = -2$

What is $\frac{dx}{dt}$ when $s = 10$?

$$x^2 + 6^2 = s^2$$

$$\frac{d}{dt}(x^2 + 6^2) = \frac{d}{dt}(s^2)$$

$$2x \frac{dx}{dt} = 2s \frac{ds}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

→ If $s = 10$,

$x^2 = 10^2 - 6^2 = 100 - 36$

$x^2 = 64$

$x = 8$

when $s = 10$ $\quad \frac{dx}{dt} = \frac{10}{8}(-2) = \frac{-10}{5} = \frac{-5}{2} = -2.5$

The speed of the boat is then 2.5 ft/sec when $s = 10$