Part A: Multiple Choice

1. A manufacturer determines that the profit P (in dollars) made from selling x units of a certain item is given by $P = 0.0003x^3 + 6x$. Find the marginal profit for a production level of 76 units.

(A) \$11.20 **(B)** \$11.40 **(C)** \$12.00 **(D)** \$ 11.05 **(E)** None of the above.

Solution: **A** $\frac{dP}{dx} = 0.0009x^2 + 6 \Rightarrow \left.\frac{dP}{dx}\right|_{x=76} = 0.0009(76)^2 + 6 = 11.20$

2. Find the equation of the tangent line to the graph of $f(x) = 2(x^2 - 1)^3$ at the point $(2, f(2))$.

(A) $y = 2x + 54$ **(B)** $y = 216x - 378$ **(C)** $y = 216x + 54$ **(D)** $y = 2x - 378$

Solution: **B** $f'(x) = 6(x^2 - 1)^2(2x) = 12x(x^2 - 1)^2 \Rightarrow f'(2) = 12(2)((2)^2 - 1)^2 = 216$, the slope $f(2) = 2(2^2 - 1)^3 = 54$. So the slope is 216 and the point is $(2, 54)$. The equation is therefore:
$y - 54 = 216(x - 2) \Rightarrow y - 54 = 216x - 432 \Rightarrow y = 216x - 378$

3. Find $f'''(-2)$ for $f(x) = \sqrt{-x-1}$.

(A) $-\frac{1}{4}$ **(B)** $-\frac{1}{8}$ **(C)** $-\frac{3}{4}$ **(D)** $-\frac{3}{8}$ **(E)** None of the above.

4. Find $\frac{dy}{dx}$ if $9xy + 6y^4 = 9x^5 + 5y$.

(A) $\frac{dy}{dx} = \frac{45x^4 - 9x}{24y^3 - 5}$ **(B)** $\frac{dy}{dx} = \frac{45x^4 - 9x + 5}{9x + 24y^3 - 5}$ **(C)** $\frac{dy}{dx} = \frac{45x^4 - 9y}{9x + 24y^3 - 5}$

(D) $\frac{dy}{dx} = \frac{45x^4 - 9y + 5}{9x + 24y^4}$ **(E)** None of the above.

5. Find $\frac{dy}{dt}$ at point $(2, -1)$ if $x^3 + y^2 = 9$ and $\frac{dx}{dt} = \frac{1}{2}$.

(A) -3 **(B)** $\frac{1}{6}$ **(C)** $\frac{13}{4}$ **(D)** 3 **(E)** None of the above.

6. Determine whether $y = -x^2 + 2x$ is increasing or decreasing at the point $(2, 0)$.

(A) Decreasing **(B)** Increasing **(C)** Turning **(D)** Not enough information

(E) None of the above.

7. Find the second derivative of $f(x) = \frac{3}{4x^2}$.

(A) $-\frac{3}{4x^4}$ **(B)** $\frac{9}{2x^4}$ **(C)** $\frac{9}{2}x$ **(D)** $-\frac{9}{2x}$ **(E)** None of the above.

8. Find the slope of the tangent line to the graph of $3x^2 - 2y + 5 = 0$ at the point $(1, 4)$.

(A) $-\frac{5}{6}$ **(B)** -3 **(C)** 0 **(D)** 3 **(E)** None of the above.

The following word problem relates to numbers 9 and 10:

A toy manufacturer has determined that the monthly demand for its best-selling toy is given by $p = \frac{40000 - x}{4000}$, where p is the price of the toy in dollars and x is the estimated number that can be sold for that price.

9. Find the revenue function R.

(A) $R = \frac{40000 - 2x}{4000}$ **(B)** $R = \frac{40000x - x^2}{4000}$ **(C)** $R = \frac{40000x^2 - x}{4000}$

(D) $R = \frac{40000x^2 - 2x}{4000}$ **(E)** None of the above.

10. Find the marginal revenue per toy when $x = 10000$?

(A) \$5.00 **(B)** \$5.25 **(C)** \$7.13 **(D)** \$7.50 **(E)** None of the above.

Part B: Free Response (Show all work **CLEARLY**!!!)
Where applicable, leave answers as fractions in simplest form. Write your answer on the line provided

Question 11

(a) Find the third derivative of the function $g(t)=\dfrac{4}{(t+2)^2}$**.**

Solution: $g(t)=\dfrac{4}{(t+2)^2}=4(t+2)^{-2}$

$$g'(t)=(-2)\cdot 4(t+2)^{-3}(1)=-8(t+2)^{-3}$$
$$g''(t)=(-3)\cdot -8(t+2)^{-4}(1)=24(t+2)^{-4}$$
$$g'''(t)=(-4)\cdot 24(t+2)^{-5}(1)=\underline{-96(t+2)^{-5}=-\frac{96}{(t+2)^5}}$$

(b) Given that $f''(x)=\dfrac{2x-2}{x}$, find $f'''(x)$

Solution: $f''(x)=\dfrac{2x-2}{x}=\dfrac{u(x)}{v(x)}$. So $f'''(x)=\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$

$u(x)=2x-2 \Rightarrow u'(x)=2$ and $v(x)=x \Rightarrow v'(x)=1$

$$f'''(x)=\frac{2x-(2x-2)(1)}{x^2}=\frac{2x-2x+2}{x^2}=\frac{2}{x^2}$$

OR

$f''(x)=\dfrac{2x-2}{x}=\dfrac{2x}{x}-\dfrac{2}{x}=2-2x^{-1}$. So $f'''(x)=0-(-1)2x^{-2}=2x^{-2}=\dfrac{2}{x^2}$

Question 12

Find the equation of the tangent line to the graph of $y^2-2x=xy$ **at the point** $(1,2)$**. Give the answer in slope-intercept form.**

Solution: To find the equation of the tangent line we need the slope $\left(\dfrac{dy}{dx}\right)$ and a point, which is $(1,2)$. We obtain the slope by differentiating both sides with respect to x and solving for $\dfrac{dy}{dx}$:

Note: Let $u(x)=x$ and $v(x)=y$. So $u'(x)=1$ and $v'(x)=\dfrac{dy}{dx}$. So $\dfrac{d}{dx}[xy]=(1)y+x\dfrac{dy}{dx}$ as substituted below:

$$\frac{d}{dx}[y^2]-\frac{d}{dx}[2x]=\frac{d}{dx}[xy] \quad \Rightarrow 2y\frac{dy}{dx}-2=u'v+uv' \quad \Rightarrow 2y\frac{dy}{dx}-2=(1)y+x\frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx}-x\frac{dy}{dx}=y+2 \quad \Rightarrow \frac{dy}{dx}(2y-x)=y+2 \quad \Rightarrow \frac{dy}{dx}=\frac{y+2}{(2y-x)} \quad \Rightarrow \left.\frac{dy}{dx}\right|_{(1,\,2)}=\frac{2+2}{(2(2)-1)}=\frac{4}{3}$$

So the slope is $m=\dfrac{4}{3}$, and the point is $(1,2)$. The equation of the tangent line is therefore:

$$y-y_1=m(x-x_1) \quad \Rightarrow y-2=\frac{4}{3}(x-1) \quad \Rightarrow y-2=\frac{4}{3}x-\frac{4}{3} \quad \Rightarrow y=\frac{4}{3}x-\frac{4}{3}+2$$

So $\underline{y=\dfrac{4}{3}x+\dfrac{2}{3}}$

Question 13

A spherical balloon is inflated with gas at a rate of 20 cubic feet per minute. How fast is the radius of the balloon changing at the instant that the radius is 2 feet? Give the appropriate unit of measurement in your answer.

Note: *The volume of a sphere is given by* $V = \frac{4}{3}\pi r^3$

Solution: We want to find the rate of change of the radius, which is $\frac{dr}{dt}$. So we differentiate both sides with respect to t: $\frac{dV}{dt} = \left(\frac{4}{3}\right)(3)\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

The rate of change of the volume is $\frac{dV}{dt} = 20$, and the radius is $r = 2$. So the rate of change of the radius is:

$$20 = 4\pi\left(2^2\right)\frac{dr}{dt} \quad \Rightarrow 20 = 16\pi\frac{dr}{dt} \quad \Rightarrow \frac{dr}{dt} = \frac{20}{16\pi} = 0.398\ ft/\text{min}$$

Question 14

The monthly demand function for x newspapers at a newsstand is $p = 5 - 0.001x$ and the monthly cost function is $C = 35 + 1.5x$.

(a) Find the monthly revenue ***R*** as a function of x.

Solution: $R = px = (5 - 0.001x)x = 5x - 0.001x^2$

(b) Find the monthly profit ***P*** as a function of x.

Solution: $P = R - C = 5x - 0.001x^2 - (35 + 1.5x) \quad \Rightarrow P = -0.001x^2 + 3.5x - 35$

P = ______________________

(c) *(i)* Find the marginal revenue as a function of x.

Solution: Marginal revenue is given by: $\frac{dR}{dx} = 5 - 0.002x$

(ii) Find the marginal profit as a function of x.

Solution: Marginal profit is given by: $\frac{dP}{dx} = -0.002x + 3.5$

(d) Find the profit and the marginal profit for each of the following levels of newspaper production:

(i) 600

$P = -0.001(600)^2 + 3.5(600) - 35 = 1705$

$\frac{dP}{dx} = -0.002(600) + 3.5 = 2.3$

Profit = $1705 **Marginal Profit** = $2.30 per copy

(ii) 1200

$P = -0.001(1200)^2 + 3.5(1200) - 35 = 2725$

$\frac{dP}{dx} = -0.002(1200) + 3.5 = 1.1$

Profit = $2725 **Marginal Profit** = $1.10 per copy

(iii) 2400

$P = -0.001(2400)^2 + 3.5(2400) - 35 = 2605$

$\frac{dP}{dx} = -0.002(2400) + 3.5 = -1.3$

Profit = $2605 **Marginal Profit** = – $1.30 per copy

(iv) 3000

$P = -0.001(3000)^2 + 3.5(3000) - 35 = 1465$

$\frac{dP}{dx} = -0.002(3000) + 3.5 = -2.5$

Profit = $1465 **Marginal Profit** = –$2.50 per copy