Part A: Multiple Choice

1. A manufacturer determines that the profit P (in dollars) made from selling x units of a certain item is given by
$P = 0.0003x^3 + 6x$.
Find the marginal profit for a production level of 76 units.

(A) \$11.20 **(B)** \$11.40 **(C)** \$12.00 **(D)** \$11.05 **(E)** None of the above

Solution: **(A)**

$$\text{Marginal Profit} = \left.\frac{dP}{dx}\right|_{x=76} = \left.(0.0009x^2 + 6)\right|_{x=76} = 0.0009(76^2) + 6 = 11.1984 \approx 11.20$$

2. Find the equation of the tangent line to $f(x) = 2(x^2 - 1)^3$ at the point $(2, f(2))$.

(A) $y = 2x + 54$ **(B)** $y = 216x - 378$ **(C)** $y = 216x + 54$

(D) $y = 2x - 378$ **(E)** None of the above

Solution: **(B)**

Point is $\left(2, 2(2^2 - 1)^3\right) = (2, 54)$; $f'(x) = 6(x^2 - 1)^2(2x) = 12x(x^2 - 1)^2$

So slope is $f'(2) = 12(2)(2^2 - 1)^2 = 216$

Equation of tangent line: $y - 54 = 216(x - 2)$ $\Rightarrow y = 216x - 378$

3. Find $f'''(-2)$ for $f(x) = \sqrt{-x-1}$.

(A) $-\frac{1}{4}$ **(B)** $-\frac{1}{8}$ **(C)** $-\frac{3}{4}$ **(D)** $-\frac{3}{8}$ **(E)** None of the above

Solution: **(D)**

$f(x) = \sqrt{-x-1} = (-x-1)^{1/2}$ $\Rightarrow f'(x) = -\frac{1}{2}(-x-1)^{-1/2}$

$f''(x) = -\frac{1}{4}(-x-1)^{-3/2}$ $\Rightarrow f'''(x) = -\frac{3}{8}(-x-1)^{-5/2}$

$f'''(-2) = -\frac{3}{8}(-(-2)-1)^{-5/2} = -\frac{3}{8}$

Name: __Solutions____

4. Find $\frac{dy}{dx}$ if $9xy + 6y^4 = 9x^5 + 5y$

(A) $\frac{dy}{dx} = \frac{45x^4 - 9x}{24y^3 - 5}$ **(B)** $\frac{dy}{dx} = \frac{45x^4 - 9x + 5}{9x + 24y^3 - 5}$ **(C)** $\frac{dy}{dx} = \frac{45x^4 - 9y}{9x + 24y^3 - 5}$

(D) $\frac{dy}{dx} = \frac{45x^5 - 9y + 5}{9x + 24y^4}$ **(E)** None of the above

Solution: **(C)**

$\frac{d}{dx}[9xy] + \frac{d}{dx}[6y^4] = \frac{d}{dx}[9x^5] + \frac{d}{dx}[5y]$

$\left[\frac{d}{dx}[9x] \cdot y + 9x\frac{d}{dx}[y]\right] + 24y^3\frac{dy}{dx} = 45x^4 + 5\frac{dy}{dx}$

$9y + 9x\frac{dy}{dx} + 24y^3\frac{dy}{dx} = 45x^4 + 5\frac{dy}{dx} \quad \Rightarrow 9x\frac{dy}{dx} + 24y^3\frac{dy}{dx} - 5\frac{dy}{dx} = 45x^4 - 9y$

$\frac{dy}{dx}(9x + 24y^3 - 5) = 45x^4 - 9y \quad \Rightarrow \frac{dy}{dx} = \frac{45x^4 - 9y}{9x + 24y^3 - 5}$

5. Find $\frac{dy}{dt}$ at point $(2,-1)$ if $x^3 + y^2 = 9$ and $\frac{dx}{dt} = \frac{1}{2}$

(A) -3 **(B)** $\frac{1}{6}$ **(C)** $\frac{13}{4}$ **(D)** 3 **(E)** None of the above

Solution: **(D)**

$\frac{d}{dt}[x^3] + \frac{d}{dt}[y^2] = \frac{d}{dt}[9] \quad \Rightarrow 3x^2\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$3(2^2)\left(\frac{1}{2}\right) + 2(-1)\frac{dy}{dt} = 0 \quad \Rightarrow -2\frac{dy}{dt} = -6 \quad \Rightarrow \frac{dy}{dt} = 3$

6. Determine whether $y = -x^2 + 2x$ is increasing, decreasing or turning at point $(2,0)$.

(A) Decreasing **(B)** Increasing **(C)** Turning

(D) Not enough information **(E)** None of the above

Solution: **(A)**

$\frac{dy}{dx} = -2x + 2 \quad \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = -2(2) + 2 = -2 < 0$ So decreasing

7. Identify the intervals on which the function $y = (x-2)^2 + 1$ is increasing and decreasing.

(A) Increasing on $(-1,\infty)$; Decreasing on $(-\infty,-1)$

(B) Increasing on $(1,\infty)$; Decreasing on $(-\infty,1)$

(C) Increasing on $(-\infty,2)$; Decreasing on $(2,\infty)$

(D) Increasing on $(2,\infty)$; Decreasing on $(-\infty,2)$

Solution: **(D)**

$\frac{dy}{dx} = 2(x-2)$. When increasing, $\frac{dy}{dx} > 0$; when decreasing, $\frac{dy}{dx} < 0$

Increasing: $2(x-2) > 0 \quad \Rightarrow x > 2$; Decreasing: $2(x-2) < 0 \quad \Rightarrow x < 2$

Name: __Solutions____

8. Find the average rate of change of $f(x) = -7 + 6x^{-1}$ on the interval $[2, 6]$.

(A) $-\frac{3}{8}$ **(B)** $-\frac{1}{2}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** None of the above

Solution: **(B)**

Average rate of change $= \dfrac{f(x_2) - f(x_2)}{x_2 - x_1} = \dfrac{f(6) - f(2)}{6 - 2} = \dfrac{\left(-7 + 6(6^{-1})\right) - \left(-7 + 6(2^{-1})\right)}{4}$

$= \dfrac{-6 - (-4)}{4} = \dfrac{-2}{4} = -\dfrac{1}{2}$

The following word problem relates to numbers 9 and 10:

A toy manufacturer has determined that the monthly demand for its best-selling toy is given by $p = \frac{40\,000 - x}{4000}$, where p is the price of the toy in dollars and x is the estimated number that can be sold for that price.

9. Find the revenue function R.

(A) $R = \dfrac{40\,000 - 2x}{4000}$ **(B)** $R = \dfrac{40\,000x - x^2}{4000}$ **(C)** $R = \dfrac{40\,000x^2 - x}{4000}$

(D) $R = \dfrac{40\,000x^2 - 2x}{4000}$ **(E)** None of the above.

Solution: **(B)**

$R = px = \left(\dfrac{40\,000 - x}{4000}\right)x = \dfrac{40\,000x - x^2}{4000}$

10. Find the marginal revenue per toy when $x = 10000$

(A) \$5.00 **(B)** \$5.25 **(C)** \$7.13 **(D)** \$7.50 **(E)** None of the above.

Solution: **(A)**

Marginal Revenue $= \left.\dfrac{dR}{dx}\right|_{x=10000} = \left.\dfrac{40\,000 - 2x}{4000}\right|_{x=10000} = \dfrac{40\,000 - 2(10000)}{4000} = 5$

Name: __Solutions____

Part B: Free Response (Show all work!!!)

11.

a. Find the third derivative of the function $g(t) = -\frac{4}{(t+2)^2}$.

Solution:

$g(t) = -\frac{4}{(t+2)^2} = -4(t+2)^{-2} \quad \Rightarrow g'(t) = 8(t+2)^{-3}$

$\Rightarrow g''(t) = -24(t+2)^{-4} \quad \Rightarrow g'''(t) = 96(t+2)^{-5} = \frac{96}{(t+2)^5}$

b. Given that $f''(x) = \frac{2x-2}{x}$, find $f'''(4)$.

Solution:

$f''(x) = \frac{2x-2}{x} = \frac{u(x)}{v(x)}; \quad u(x) = 2x-2 \Rightarrow u'(x) = 2; \quad v(x) = x \Rightarrow v'(x) = 1$

$f'''(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2} = \frac{2x - (2x-2)(1)}{x^2} = \frac{2}{x^2}$

$f'''(4) = \frac{2}{4^2} = \frac{1}{8}$

12. Find the equation (in slope-intercept form) of the tangent line to the graph of $y^2 - 2x = xy$ at the point $(1, 2)$.

Solution:

$\frac{d}{dx}[y^2] - \frac{d}{dx}[2x] = \frac{d}{dx}[xy] \quad \Rightarrow 2y\frac{dy}{dx} - 2 = \left[\frac{d}{dx}[x] \cdot y + x\frac{dy}{dx}\right]$

$2y\frac{dy}{dx} - 2 = y + x\frac{dy}{dx} \quad \Rightarrow 2y\frac{dy}{dx} - x\frac{dy}{dx} = y + 2 \quad \Rightarrow (2(2)-1)\frac{dy}{dx} = 2+2$

Slope $= \frac{4}{3} \quad \Rightarrow$ Equation of tangent line: $y - 2 = \frac{4}{3}(x-1)$

$\Rightarrow y = \frac{4}{3}x + \frac{2}{3}$

13. A point is moving along the graph of $2x^2 + xy = 21$. At the point $(3, 1)$, the x –coordinate is moving at a rate of 6 units per second. At what rate is the y –coordinate moving at this point?

Solution:

The rate at which the y –coordinate is moving is $\frac{dy}{dt}$, so we differentiate both sides of the equation with respect to t. Note also that the rate of change of the x –coordinate is $\frac{dx}{dt} = 6$

$\frac{d}{dt}[2x^2] + \frac{d}{dt}[xy] = \frac{d}{dt}[21] \quad \Rightarrow 4x\frac{dx}{dt} + \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = 0$

Substituting the values of $\frac{dx}{dt} = 6, \; x = 3, \; y = 1$ gives:

$4(3)(6) + (6)(1) + (3)\frac{dy}{dt} = 0 \quad \Rightarrow 78 + 3 \cdot \frac{dy}{dt} = 0 \quad \Rightarrow \frac{dy}{dt} = -26$

Name: __Solutions____

14. Given the function:

$$g(x) = 6x^3 - 15x^2 + 12x$$

a. Determine all critical numbers of $g(x)$.

Solution:

Since $g(x)$ is a polynomial, then there is no point of singularity, and the natural domain is all Real Numbers. Thus there are no endpoints (since there was no interval stated). Thus, the only candidates for critical numbers are turning points; that is, x –values for which $g'(x) = 0$:

$g'(x) = 18x^2 - 30x + 12 = 0 \quad \Rightarrow 3x^2 - 5x + 2 = 0 \quad \Rightarrow (3x-2)(x-1) = 0$

$\Rightarrow 3x - 2 = 0$ or $x - 1 = 0 \quad \Rightarrow x = \frac{2}{3}$ or $x = 1$

So critical numbers are $x = \frac{2}{3}$, $x = 1$

b. Determine the intervals on which the following occur:

i. $g(x)$ is increasing

Solution:

$g'(x) = 18x^2 - 30x + 12 > 0 \quad \Rightarrow 3x^2 - 5x + 2 > 0 \quad \Rightarrow (3x-2)(x-1) > 0$

$\Rightarrow 3x - 2 > 0$ and $x - 1 > 0$; OR $3x - 2 < 0$ and $x - 1 < 0$

$\Rightarrow x > \frac{2}{3}$ and $x > 1 \Rightarrow x > 1$; OR $x < \frac{2}{3}$ and $x < 1 \Rightarrow x < \frac{2}{3}$

So increasing on $\left(-\infty, \frac{2}{3}\right) \cup (1, \infty)$

ii. $g(x)$ is decreasing

Solution:

$g'(x) = 18x^2 - 30x + 12 < 0 \quad \Rightarrow 3x^2 - 5x + 2 < 0 \quad \Rightarrow (3x-2)(x-1) < 0$

$\Rightarrow 3x - 2 > 0$ and $x - 1 < 0$; OR $3x - 2 < 0$ and $x - 1 > 0$

$\Rightarrow x > \frac{2}{3}$ and $x < 1 \Rightarrow x > 1$; OR $x < \frac{2}{3}$ and $x > 1$ (Impossible)

So decreasing on $\left(\frac{2}{3}, 1\right)$