Solutions*

1. For what values of B is the following function continuous at $x = 1$?

$$f(x) = \begin{cases} 3x^3 - x^2 - Bx, & \text{if } x > 1; \\ Bx - 2, & \text{if } x \leq 1. \end{cases}$$

Solution to 1. The limit from the left (at $x = 1$), $\lim_{x\to 1^-} f(x)$ is $3 \cdot (1)^3 - 1^2 - B \cdot 1 = 2 - B$, whereas the limit from the right $\lim_{x\to 1^+} f(x)$ is $B \cdot 1 - 2 = B - 2$. In order for $f(x)$ to be continuous at $x = 1$ we set these equal and solve. $2 - B = B - 2$ getting $B = 2$.

Answer to 1. $B = 2$

2. Find the equation of the tangent line to the curve $y = \frac{x-1}{x+1}$ at each point where the tangent is parallel to the line $x - 2y = 2$.

Solution to 2. The line $x - 2y = 2$, can be written as $y = x/2 - 1$, hence its slope is $1/2$. Now the slope of the tangent to the curve $y = \frac{x-1}{x+1}$ is

$$\frac{dy}{dx} = \frac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2} = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2} \quad .$$

Setting this equal to $1/2$ yields that the points we are interested in satisfy

$$\frac{2}{(x+1)^2} = \frac{1}{2} \quad .$$

Solving this we get $(x+1)^2 - 4 = 0$ hence, $x^2 + 2x - 3 = (x+3)(x-1) = 0$, whose roots are $x = -3$ and $x = 1$. Plugging-in into $y = \frac{x-1}{x+1}$, gives, when $x = -3$ $y = 2$, and when $x = 1$, $y = 0$. So the points we need are $(-3, 2)$ and $(1, 0)$.

Answer to 2. The tangent at the point $(-3, 2)$ is $(y - 2) = (1/2)(x + 3)$, which is $x - 2y = -7$, and the tangent at the point $(1, 0)$ is $(y - 0) = (1/2)(x - 1)$, which is $x - 2y = 1$.

3. Find the equation of the tangent line to the curve defined by the equation $\ln(xy) + 2x - y + 1 = 0$ at the point $(\frac{1}{2}, 2)$.

Solution to 3. First simplify to get

$$\ln x + \ln y + 2x - y + 1 = 0 \quad .$$

By implicit differentiation

$$\frac{1}{x} + \frac{y'}{y} + 2 - y' = 0 \quad .$$

Now plug-in $x = 1/2$ and $y = 2$, to get

$$\frac{1}{1/2} + \frac{y'}{2} + 2 - y' = 0 \quad ,$$

getting $4 = y'/2$, and so the slope y' (at the designated point) is 8. The equation of the tangent line at the designated point $(\frac{1}{2}, 2)$ is $(y-2) = 8(x - \frac{1}{2})$, which is $y = 8x - 2$.

Answer to 3. $y = 8x - 2$

4. Find the following limits:

$$a) \lim_{x \to -1} (x^2 - 2x + 1) \quad b) \lim_{x \to \infty} \frac{3x^2 - 7}{\sqrt{x^2 + 2}} \quad c) \lim_{x \to 0} \frac{\sin 3x}{2 \sin 5x}$$

$$d) \lim_{x \to \infty} \frac{3e^x + 4e^{-x}}{5e^x + 4e^{-x}} \quad e) \lim_{x \to 8^-} \frac{|x - 8|}{x - 8} \quad .$$

Solution to 4.

(a) Plugging in we get $(-1)^2 - 2(-1) + 1 = 4$.

Answer to 4a) 4

(b) "forgetting about the little ones" we get

$$\lim_{x \to \infty} \frac{3x^2 - 7}{\sqrt{x^2 + 2}} = \lim_{x \to \infty} \frac{3x^2}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{3x^2}{x} = \lim_{x \to \infty} 3x = \infty \quad .$$

Answer to 4b) ∞

(c) When the argument of sin goes to 0, you can replace $\sin w$ by w, hence

$$\lim_{x \to 0} \frac{\sin 3x}{2 \sin 5x} = \lim_{x \to 0} \frac{3x}{2(5x)} = \lim_{x \to 0} \frac{3}{10} = \frac{3}{10} \quad .$$

Answer to 4c) $3/10$

d) It is better to replace e^x by another variable, let's call it y.

$$\lim_{x \to \infty} \frac{3e^x + 4e^{-x}}{5e^x + 4e^{-x}} = \lim_{y \to \infty} \frac{3y + 4/y}{5y + 4/y}$$

Now 'forgetting about the little ones' this equals

$$\lim_{y \to \infty} \frac{3y}{5y} = \frac{3}{5} \quad .$$

Answer to 4d) $3/5$

e) Recall that $|z| = -z$ for $z < 0$, so $|x-8| = -(x-8)$ for $x < 8$. We have

$$\lim_{x\to 8^-} \frac{|x-8|}{x-8} = \lim_{x\to 8^-} \frac{-(x-8)}{x-8} = \lim_{x\to 8^-} (-1) = -1 \quad .$$

Answer to 4e) -1

5. Find the following limits

$$a) \quad \lim_{x\to 0^+} \sqrt{x}\ln x \qquad b) \quad \lim_{x\to 0^+} x^{\sin x} \qquad c) \quad \lim_{x\to\infty} \frac{(\ln x)^2}{x} \quad .$$

Solution to 5 a) We use l'Hôpital:

$$\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(x^{-1/2})'} = \lim_{x\to 0^+} \frac{1/x}{(-1/2)x^{-3/2}} =$$

$$\lim_{x\to 0^+} \frac{x^{1/2}}{(-1/2)} = \lim_{x\to 0^+} -2\sqrt{x} = \lim_{x\to 0^+} -2\sqrt{0} = 0 \quad .$$

Answer to 5a) 0

b) We must first take the ln, getting $\ln(x^{\sin x}) = \sin x \ln x$, and writing it as a quotient we get $\frac{\ln x}{\csc x}$. Taking the limit of this we have:

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{(\ln x)'}{(\csc x)'} = \lim_{x\to 0^+} \frac{1/x}{-\cot x \csc x} =$$

$$\lim_{x\to 0^+} \frac{\tan x \sin x}{x} = \lim_{x\to 0^+} \frac{\sin x \sin x}{(\cos x)x} = \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0 \quad .$$

Now the limit of the original is the exponential of this

Answer to 5b) $e^0 = 1$

Shortcut: When x is near 0, $\sin x$ can be replaced by x, so we get the simpler limit $\lim_{x\to 0^+} x^x$ whose ln is $x\ln x$, and the limit

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(x^{-1})'} =$$

$$\lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0 \quad .$$

and we get quicker **Answer** $e^0 = 1$.

c) By l'Hôpital again:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{((\ln x)^2)'}{x'} = \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} \quad .$$

Using l'Hôpital once again, we have that this equals

$$2\lim_{x\to\infty}\frac{(\ln x)'}{x'} = 2\lim_{x\to\infty}\frac{1/x}{1} = 2\lim_{x\to\infty}\frac{1}{x} = 0 \quad .$$

Answer to 5c) 0

6. Find y' in each case. Do not simplify your answer.

$$a)\quad y = x^7 - 3x + 6 - \frac{1}{x^4} \quad b)\quad y = x^{5x}\sin(x^2) \quad c)\quad y = \frac{2\tan x}{\sqrt{1-x^2}}$$

$$d)\quad x^4y + 5y^6x^3 = 8 \quad e)\quad xe^{xy+3y} = y \quad f)\quad f(x) = (1+2x)^{1/x} \quad .$$

Solution to 6.

a)

$$y' = (x^7 - 3x + 6 - \frac{1}{x^4})' = (x^7 - 3x + 6 - x^{-4})' = 7x^6 - 3 - (-4)x^{-5} = 7x^6 - 3 + \frac{4}{x^5} \quad .$$

Answer to 6a) $7x^6 - 3 + \frac{4}{x^5}$

b)

$$y' = (x^{5x}\sin(x^2))' = (x^{5x})'\sin(x^2) + x^{5x}(\sin(x^2))' = (e^{5x\ln x})'\sin(x^2) + x^{5x}(\sin(x^2))' =$$

$$(5x\ln x)'(e^{5x\ln x})\sin(x^2) + x^{5x}(2x)\cos(x^2) =$$

$$(5\ln x + 5)(e^{5x\ln x})\sin(x^2) + x^{5x}(2x)\cos(x^2) =$$

$$(5\ln x + 5)x^{5x}\sin(x^2) + x^{5x}(2x)\cos(x^2) \quad .$$

Answer to 6b) $(5\ln x + 5)x^{5x}\sin(x^2) + x^{5x}(2x)\cos(x^2)$

c)

$$y' = (\frac{2\tan x}{\sqrt{1-x^2}})' = \frac{(2\tan x)'\sqrt{1-x^2} - (2\tan x)((1-x^2)^{1/2})'}{(\sqrt{1-x^2})^2} =$$

$$\frac{2\sqrt{1-x^2}\sec^2 x - (2\tan x)(1/2)(1-x^2)^{-1/2}\cdot(-2x)}{1-x^2} =$$

$$\frac{2\sqrt{1-x^2}\sec^2 x + 2x(1-x^2)^{-1/2}\tan x}{1-x^2} = \frac{2\sec^2 x}{\sqrt{1-x^2}} + \frac{2x\tan x}{\sqrt{1-x^2}^3}$$

Answer to 6c) $\frac{2\sec^2 x}{\sqrt{1-x^2}} + \frac{2x\tan x}{\sqrt{1-x^2}^3}$

d)

$$(x^4y + 5y^6x^3)' = 0 \quad ,$$

so

$$(x^4)'y + x^4y' + 5y^6(x^3)' + 5(y^6)'x^3 = 0 \quad ,$$

$$4x^3y + x^4y' + 5y^6(3x^2) + 5(6y^5y')x^3 = 0 \quad ,$$

$$4x^3y + x^4y' + 15y^6x^2 + 30y^5y'x^3 = 0 \quad .$$

Moving all the terms not involving y' to the right gives

$$x^4y' + 30x^3y^5y' = -4x^3y - 15y^6x^2 \quad .$$

$$(x^4 + 30x^3y^5)y' = -4x^3y - 15y^6x^2 \quad .$$

Hence

$$y' = \frac{-4x^3y - 15y^6x^2}{x^4 + 30x^3y^5}$$

Since we are not supposed to simplify, we have

Answer to 6d) $y' = \frac{-4x^3y-15y^6x^2}{x^4+30x^3y^5}$

e)

$$(xe^{xy+3y})' = y' \quad ,$$

$$x'e^{xy+3y} + x(e^{xy+3y})' = y' \quad ,$$

$$e^{xy+3y} + xe^{xy+3y} \cdot (xy + 3y)' = y' \quad ,$$

$$e^{xy+3y} + xe^{xy+3y} \cdot (x'y + xy' + 3y') = y' \quad ,$$

$$e^{xy+3y} + xe^{xy+3y} \cdot (y + (x + 3)y') = y' \quad ,$$

$$e^{xy+3y} + xye^{xy+3y} + (x + 3)xy'e^{xy+3y} = y'$$

$$(1 + xy)e^{xy+3y} = y'(1 - x(x + 3)e^{xy+3y})$$

Hence

$$y' = \frac{e^{xy+3y}(1 + xy)}{1 - x(x + 3)e^{xy+3y}} \quad .$$

Answer to 6e) $y' = \frac{e^{xy+3y}(1+xy)}{1-x(x+3)e^{xy+3y}}$

f)

$$(\ln f(x))' = (\frac{\ln(1 + 2x)}{x})'$$

$$\frac{f'(x)}{f(x)} = \frac{(\ln(1 + 2x))'x - \ln(1 + 2x) \cdot x'}{x^2} = \frac{(2x/(1 + 2x)) - \ln(1 + 2x)}{x^2}$$

Hence

$$f'(x) = f(x)\frac{(2x/(1 + 2x) - \ln(1 + 2x)}{x^2} = (1 + 2x)^{1/x}\frac{2x/(1 + 2x) - \ln(1 + 2x)}{x^2} \quad .$$

Answer to 6f) $(1+2x)^{1/x}\frac{2x/(1+2x)-\ln(1+2x)}{x^2}$

7. State the formal definition of the derivative of the function $f(x)$. Use the definition to calculate $f'(x)$ for $f(x)=\sqrt{3-5x}$.

Solution to 7.

The formal definition of the derivative of a function $f(x)$ is

$$f'(x)=\lim_{s\to x}\frac{f(s)-f(x)}{s-x} \quad .$$

Note: There is another version with h and $\lim_{h\to 0}$, but for *this* problem it is more convenient to use this one.

$$f'(x)=\lim_{s\to x}\frac{\sqrt{3-5s}-\sqrt{3-5x}}{s-x}=\lim_{s\to x}\frac{(\sqrt{3-5s}-\sqrt{3-5x})(\sqrt{3-5s}+\sqrt{3-5x})}{(s-x)(\sqrt{3-5s}+\sqrt{3-5x})}$$

$$=\lim_{s\to x}\frac{(\sqrt{3-5s})^2-(\sqrt{3-5x})^2}{(s-x)(\sqrt{3-5s}+\sqrt{3-5x})}=\lim_{s\to x}\frac{(3-5s)-(3-5x)}{(s-x)(\sqrt{3-5s}+\sqrt{3-5x})}$$

$$=\lim_{s\to x}\frac{-5(s-x)}{(s-x)(\sqrt{3-5s}+\sqrt{3-5x})}=$$

$$\lim_{s\to x}\frac{-5}{\sqrt{3-5s}+\sqrt{3-5x}}=\frac{-5}{\sqrt{3-5x}+\sqrt{3-5x}}=\frac{-5}{2\sqrt{3-5x}} \quad .$$

Answer to 7. $\frac{-5}{2\sqrt{3-5x}}$ (but it had to be derived from the definition!).

8. Suppose that $S(x)=\sqrt{x}$ for $x\geq 0$, and let f and g be differentiable functions about which the following is known

$$f(3)=2, \quad f'(3)=7, \quad g(3)=4, \quad g'(3)=5 \quad .$$

Compute the following

$$(f+g)'(3) \quad (f\cdot g)'(3), \quad \left(\frac{f}{g}\right)'(3), \quad (S\circ g)'(3), \quad \left(\frac{f\cdot g}{f-g}\right)'(3) \quad .$$

Solution to 8.

$$(f+g)'(3)=f'(3)+g'(3)=7+5=12 \quad .$$

$$(f\cdot g)'(3)=f'(3)\cdot g(3)+f(3)\cdot g'(3)=7\cdot 4+2\cdot 5=38 \quad .$$

$$\left(\frac{f}{g}\right)'(3)=\frac{f'(3)g(3)-f(3)g'(3)}{g(3)^2}=\frac{7\cdot 4-2\cdot 5}{4^2}=\frac{18}{16}=\frac{9}{8} \quad .$$

$$(S\circ g)'(3)=S'(g(3))g'(3)=S'(4)\cdot 5=5\cdot(1/2)(4)^{-1/2}=5/4 \quad .$$

$$\left(\frac{f\cdot g}{f-g}\right)'(3)=$$

$$\left(\frac{(f\cdot g)'(f-g)-(f\cdot g)(f-g)'}{(f-g)^2}\right)(3)=$$
$$\left(\frac{(f'g+fg')(f-g)-(f\cdot g)(f'-g')}{(f-g)^2}\right)(3)=$$
$$\left(\frac{f^2g'-f'g^2}{(f-g)^2}\right)(3)=$$
$$\frac{f(3)^2g'(3)-f'(3)g(3)^2}{(f(3)-g(3))^2}=$$
$$\frac{2^2\cdot 5-7\cdot 4^2}{(2-4)^2}=-23 \quad .$$

Answer to 8. 12, 38, 9/8, 5/4, -23

9. Suppose $f''(x) = -3x + \cos(\pi x)$ and $f(1) = 2$ and $f'(1) = -1$. What is $f(5)$?

Solution to 9. The general antiderivative of $-3x + \cos(\pi x)$ is $f'(x) = -3x^2/2 + \sin(\pi x)/\pi + A$, where A is a general constant. To find A, we plug-in $x = 1$, and get $f'(1) = -3\cdot 1^2/2+\sin(\pi)/\pi+A = -3/2 + 0 + A$. On the other hand, in this specific problem, we are told that $f'(1) = -1$, so we get the equation $-1 = A - 3/2$, and solving for A, we get $A = 1/2$. So

$$f'(x) = -(3/2)x^2 + \sin(\pi x)/\pi + 1/2 \quad .$$

To get $f(x)$, we take the antiderivative of the above, getting $f(x) = -(1/2)x^3-\cos(\pi x)/\pi^2+x/2+B$, where B is a constant yet to be determined. Plugging-in $x = 1$ into this yields $f(1) = -1/2 - \cos(\pi)/\pi^2 + 1/2 + B = 1/\pi^2 + B$. On the other hand, in this specific problem, we are told that $f(1) = 2$, so we get the equation $2 = 1/\pi^2 + B$, so $B = 2 - 1/\pi^2$. Hence

$$f(x) = -(1/2)x^3 - \cos(\pi x)/\pi^2 + x/2 + 2 - 1/\pi^2$$

Finally, plugging-in $x = 5$, we get $f(5) = -(1/2)5^3 - \cos(5\pi)/\pi^2 + 5/2 + 2 - 1/\pi^2 = -125/2 + 1/\pi^2 + 5/2 + 2 - 1/\pi^2 = -120/2 + 2 = -60 + 2 = -58$.

Answer to 9. -58

10. A farmer with 450 feet of fencing wants to enclose the four sides of a rectangular region into four pens of equal size with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Solution to 10. Let's call the length of the side parallel to the three internal walls x, and the length of the other side (perpendicular to these walls) y.

We have to **maximize** the **area**. So the **Goal** function is xy. We also have the **constraint**: $2y + 5x = 450$. Solving for y, we get $y = (450 - 5x)/2$, and plugging this into the goal function, we express it in terms of x alone, getting $f(x) = x(450 - 5x)/2 = (450/2)x - (5/2)x^2$. To find the maximum, we take the derivative $f'(x) = 450/2 - 5x$, and then set it equal to 0. Solving $450/2 - 5x = 0$, we get $x = 45$. Plugging this into $f(x)$ we get that the maximal total area is $f(45) = 45 \cdot (450 - 5 \cdot 45)/2 = 45^2 \cdot 5/2 = 10125/2$.

Ans to 10. 10125/2.

Note: In some problems, they might ask you about the **dimensions** of the optimal rectangle. Then the answer would be $45 \times (225/2)$.

11. A ladder of length 15 feet is leaning against a wall when its base begins to slide along the floor, away from the wall. By the time the base is 12 feet away from the wall, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall at that moment? How fast is the area of the triangle formed by the ladder, the wall, and the floor changing at that time?

Solution to 11. This is a related rates problem. Let's call the height of the top of the ladder y, and the distance of the foot of the ladder from the wall x. By Pythagoras,

$$x^2 + y^2 = 15^2 \quad .$$

Taking derivatives with respect to time (using the chain rule in its abstract setting, a.k.a. implicit differentiation), we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad .$$

Right now $x = 12$ and $dx/dt = 5$. We also need y right now. Using the relation, $x^2 + y^2 = 15^2$ we plug-in $x = 12$ and solve for y: $12^2 + y^2 = 15^2$, getting $y = 9$. Substituting these values into the derived relation we get

$$2 \cdot 12 \cdot 5 + 2 \cdot 9\frac{dy}{dt} = 0 \quad ,$$

and solving for dy/dt, we get $dy/dt = -20/3$. Since the sign is negative, it is sliding **down** (which is also obvious from common sense).

Answer to the first part of 11. The top of the ladder is sliding down at a speed of 20/3 ft/sec.

The **area**, let's call it A, is given by the relation $A = xy/2$. Taking derivative with respect to time (using the product rule), we get

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) \quad ,$$

and we plug-in the above numbers, getting

$$\frac{dA}{dt} = \frac{1}{2}(12 \cdot (-20/3) + 9 \cdot 5) = -35/2 \quad ,$$

Answer to second part of 11. The area is decreasing at a rate of 35/2 square feet per second.

12. Let $f(x) = \frac{3x}{x^2-1}$. Find the domain of the function, the intervals where $f(x)$ is increasing or decreasing, maximum and minimum, the concavity and inflection points, horizontal and vertical asymptotes of the graph. Then sketch the graph of $f(x)$.

Solution to 12. Domain: $\{x|x \neq -1, 1\}$. Even though the horizontal, and vertical asymptotes are mentioned **last** in the statement of the problem, it is best to start with them, since they contain the most important information.

horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x}{x^2-1} = \lim_{x\to\infty} \frac{3x}{x^2} = 0 \quad .$$

So $y = 0$ (alias the x-axis) is a horizontal asymptote.

vertical asymptotes: Set the denominator equal to 0, we get $x = -1$ and $x = 1$.

Let's consider each vertical asymptote on its own. Plugging-in a value of x slightly less than -1, say -1.01, we get $-BIG$, so this means that $\lim_{x\to-1^-} f(x) = -\infty$, and the curve would drop down to "hell" $(-\infty)$ as it approaches the vertical asymptote $x = -1$. Plugging-in a value of x slightly bigger than -1, say $-.99$ we get that $f(-.99) = BIG$, so the curve "comes down from heavens" right after $x = -1$. Similarly the curve goes down to hell as x approaches $x = 1$, and emerges from heavens right after $x = 1$.

It is also a good idea to find the x-intercept (solve $y = 0$, getting $(0, 0)$) and the y-intercept, plugging-in $x = 0$, getting, once again $(0, 0)$. So the curve intersects the x-axis only at the origin $(0, 0)$.

Notice that we haven't taken derivatives yet, but we can already pretty much draw the function. When x is very negative, it is almost coincident with the x-axis (the horizontal Asymptote) but it is below the x-axis, since $f(x)$ is negative there, and then it is dropping down-down right before $x = -1$. By inspection you can see that the curve is **concave down** at the interval $-\infty < x \leq -1$

Then from $x = -1$ to $x = 1$ it is a steady decrease (since at $x = -1$ it is coming from $y = \infty$ dropping to $y = -\infty$ at $x = 1$. Somewhere between these, there should be an inflection point, that by symmetry is $(0, 0)$. Also for $x > 1$ it dropping from ∞ to 0. So you can draw the curve without any calculus! Also it is clear that there are no local maxima and minima.

But now we can do some calculus.

By the quotient rule

$$f'(x) = -3\frac{x^2+1}{(x^2-1)^2} \quad ,$$

this is always negative, so **intervals of increase:** none.

intervals of decrease: $(-\infty, -1)$,$(-1, 1)$, $(1, \infty)$.

To find potential (local) max or min. points, solve $f'(x) = 0$, which has no roots. Hence

max and min. points: none.

Using the quotient rule one more time, we get

$$f''(x) = \frac{6x(x^2+3)}{(x^2-1)^3} \quad .$$

The inflection point(s) are where $f''(x) = 0$. Setting the numerator equal to 0, we get $x = 0$. The y coordinate is $f(0) = 0$. So

inflection point: $(0, 0)$.

The transition points in the life of this function are $x = -1$,$x = 0$, and $x = 1$. Plugging-in random values in each interval we get

The interval $(-\infty, -1)$ is **concave down** (since. e.g. $f''(2) < 0$).

The interval $(-1, 0)$ is **concave up** (since. e.g. $f''(-\frac{1}{2}) > 0$).

The interval $(0, 1)$ is **concave down** (since. e.g. $f''(\frac{1}{2}) < 0$).

The interval $(1, \infty)$ is **concave up** (since. e.g. $f''(2) > 0$).

Answer to 12. max. and min. points: none; intervals of increase: none; intervals of decrease: $(-\infty, -1), (-1, 1), (1, \infty)$; inflection point: $(0, 0)$; intervals of concave down:$(-\infty, -1)$, $(0, 1)$; intervals of concave up:$(-1, 0)$, $(1, \infty)$; horizontal asymptote: $y = 0$; vertical asymptotes: $x = -1$ and $x = 1$.

13. Repeat the same problem for $y = \frac{x^2}{x^2+3}$.

Solution to 13.

Horizontal Asymptotes: $y = 1$. Vertical asymptote: none. By the quotient rule (do it!)

$$f'(x) = \frac{6x}{x^2+3}$$

$$f''(x) = \frac{-18(x^2-1)}{x^2+3}$$

So potential max or min.: $x = 0$. Plugging into the second derivative, we get $f''(0) = 6 > 0$ so $x = 0$ is a **minimum**. Plugging $x = 0$ into $f(x)$, we get that when $x = 0$, $y = 0$, so the **point** $(0, 0)$ is a **local minimum**, and that's the only one.

To get the inflection points solve $f''(x) = 0$, and get $x = -1$ and $x = 1$. When $x < -1$, $f''(x)$ is negative so in the interval $-\infty < x < -1$ the curve is **concave down**. In the interval $-1 < x < 1$

$f''(x)$ is positive, so the curve is **concave up**, and in the interval $1 < x < \infty$ the curve is **concave down** since $f''(x)$ is negative there.

Now draw it!

Answer to 13. Horizontal asymptote $y = 1$; Vertical asymptotes: none. local min. $(0, 0)$; inflection points: $(-1, 1/4)$ and $(1, 1/4)$; interval of decrease: $(-\infty, 0)$; interval of increase: $(0, \infty)$; intervals of concave down:$(-\infty, -1), (1, \infty)$; interval of concave up: $(-1, 1)$.

14. Sketch the graph of $f(x)$ which satisfies the following conditions $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$.

Solution to 14. First let's spell everything out w/o the annoying absolute value symbol $||$.

For $-\infty < x < -2 : f'(x) = -1$.

For $-2 < x < -1 : f'(x) > 0$.

For $-1 < x < 1 : f'(x) < 0$.

For $1 < x < 2 : f'(x) > 0$.

For $2 < x < \infty : f'(x) = -1$.

Now we are almost ready to sketch, noting that at $x = -2$ and $x = 1$ we have local minima, since the derivative changes sign from negative to positive, and at $x = -1$ and $x = 2$ we have local maxima, since $f'(x)$ changes sign from positive to negative. We also have an inflection point at $(0, 1)$ and since $f''(x) < 0$ between $x = -2$ and $x = 0$, it is concave down there, and hence between $x = 0$ and $x = 2$ it is concave up.

To actually draw a curve fitting all these specifications, draw a straight line with slope -1 coming from $-\infty$ to the point $(-2, 0)$, then draw a hat-like curve that peaks at $x = -1$ and inflects at the point $(0, 1)$ turning into a cup-like arc that bottoms at $x = 1$ and goes up to $x = 2$ and then it suddenly becomes a straight line with slope -1.

15. Find the linearization of $f(x) = \sqrt{x + 1}$ at $a = 15$ and use it to find an approximation to $\sqrt{15}$ and $\sqrt{17}$. Use the second derivative to determine whether the estimate is greater or less than the actual value.

Solution to 15. The linearization of a function $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem $f(x) = (x + 1)^{1/2}$, so

$$f'(x) = (1/2)(x + 1)^{-1/2} = \frac{1}{2\sqrt{x + 1}} \quad .$$

We have $f(a) = f(15) = \sqrt{1 + 15} = \sqrt{16} = 4$, and $f'(a) = f'(15) = \frac{1}{2\sqrt{15+1}} = 1/8$. Hence

Answer to first part of 15. $L(x) = 4 + \frac{1}{8}(x - 15)$

Now $\sqrt{15} = f(14)$, and plugging $x = 14$, we get $L(14) = 4 + \frac{1}{8}(14 - 15) = \frac{31}{8}$, hence

Answer to second part of 15. $\sqrt{15}$ is approximately $\frac{31}{8}$

Also $\sqrt{17} = f(16)$, and plugging $x = 16$, we get $L(16) = 4 + \frac{1}{8}(16 - 15) = \frac{33}{8}$, hence

Answer to third part of 15. $\sqrt{17}$ is approximately $\frac{33}{8}$

$$f''(x) = (-1/4)(x+1)^{-3/2} = \frac{-1}{4\sqrt{x+1}^3} \quad .$$

Since $f''(15) = -1/256$ is **negative**, the above approximations to $\sqrt{15}$ and $\sqrt{17}$ are **larger** than the actual value.

Answer to fourth part of 15. Both are larger than the actual values.

16. Find the derivative of the following functions:

$$a) \quad F(x) = \int_{\pi}^{x} \tan(s^2)ds \qquad b) \quad g(x) = \int_{1}^{\cos x} \sqrt{1-t^2}dt \quad .$$

Solution to 16.a) Since the upper-limit of integration is **identical** to the argument of F, (both are x), the derivative is simply the integrand with s replaced by x.

Answer to 16a) $F'(x) = \tan(x^2)$

Solution to 16b) Now the upper-limit of integration, $\cos x$ is **not** identical with the argument of g, which is x. We must first use the chain rule, multiplying top and bottom by $d\cos x$ (so to speak!).

$$g'(x) = \frac{d}{dx}\int_{1}^{\cos x} \sqrt{1-t^2}dt = \frac{d\cos x}{dx} \cdot \frac{d}{\cos x}\int_{1}^{\cos x} \sqrt{1-t^2}dt \quad .$$

The first piece is a straightforward differentiation, while the second is done exactly as in a): replace t by $\cos x$ in the integrand. Hence $F'(x) = -\sin x\sqrt{1-\cos x^2}$. Using the famous trig. identity this is $-\sin^2 x$.

Answer to 16b) $g'(x) = -\sin^2 x$

17. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$ and $2 \le f'(x) \le 5$ for all $x \in (0, 4)$. Show that $9 \le f(4) \le 21$.

Solution to 17. We must use the **Mean Value Theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad .$$

In this problem $a = 0$ and $b = 4$ so, there exists a c in $(0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4 - 0}$$

But we know that $2 \le f'(x) \le 5$ for *every* $x \in (0, 4)$, in particular for the c, whatever it is. Hence

$$2 \le \frac{f(4) - 1}{4} \le 5 \quad .$$

Multiplying by 4, we get (since 4 is positive, the inequalities are preserved) $8 \le f(4) - 1 \le 20$, and adding 1 everywhere, we get $9 \le f(4) \le 21$.

18. Let $f(x) = 3x^3 - 2x^2 + x - 1$. Show that $f(x)$ must have a real root in $[0, 1]$.

Solution to 18. The given function $f(x)$ is continuous (because it is a polynomial), Plug-in the leftmost end, 0, into $f(x)$, getting $f(0) = -1$. Plug-in the rightmost end, 1, into $f(x)$, getting $f(1) = 3 \cdot 1^3 - 2 \cdot 1^2 + 1 - 1 = 1$. Since these two numbers -1 and 1 have **opposite signs** (one is positive and the other is negative), it follows by the **intermediate value theorem**, that there must be a $c \in (0, 1)$ such that $f(c) = 0$. In other words, $f(x) = 0$ must have a real root in $[0, 1]$.

19. Show that the equation $x^{101} + x^{51} + x - 1 = 0$ has exactly one real root.

Solution to 19 Let $f(x) = x^{101} + x^{51} + x - 1$. Since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$ (because the polynomial $f(x)$ has **odd degree**), and these have **opposite signs** (like in problem 18), there must be **at least** one root in $(-\infty, \infty)$.

Taking derivative, $f'(x) = 101x^{100} + 51x^{50} + 1$. Since all the powers are even, and the coefficients are **positive** and the constant term (1) is **strictly positive**, $f'(x)$ is **strictly positive**. If there would have been more than one root, by Rolle's theorem there would have been a c such that $f'(c) = 0$, contradicting the fact that $f'(x)$ is always positive. Hence there is **exactly one real root**.

20. Use two steps of Newton's method to approximate the root of the equation $x^4 + x - 4 = 0$ in the interval $[1, 2]$.

Solution to 20, Recall that **Newton's method** is:

$$x_n = x_o - \frac{f(x_o)}{f'(x_o)} \quad ,$$

where x_0 is the **old** approximation and x_n is the **new** approximation. Implementing this here, we get $f(x) = x^4 + x - 4$, and $f'(x) = 4x^3 + 1$, so for this problem we have

$$x_n = x_o - \frac{x_o^4 + x_o - 4}{4x_o^3 + 1} \quad .$$

$f(1) = -2$ and $f(2) = 14$. Since -2 is so much closer to 0 than 14, it is reasonable to take as the **initial approximation** $x_o = 1$. We get

$$x_n = 1 - \frac{1^4 + 1 - 4}{4 \cdot 1^3 + 1} = 1 - \frac{-2}{5} = 7/5 \quad .$$

Now let $x_0 = 7/5$ and we get

$$x_n = 7/5 - \frac{(7/5)^4 + 7/5 - 4}{4 \cdot (7/5)^3 + 1} \quad .$$

Since you don't have a calculator, you leave it like this. (It turns out to be 9703/7485).

Answer to 20. $7/5 - \frac{(7/5)^4+7/5-4}{4\cdot(7/5)^3+1} = 9703/7485$

21. If a stone is thrown vertically upward from the surface of the moon with a velocity of $10m/s$, its height (in meters) after t seconds is $h(t) = 10t - t^2$.

(a) What is the velocity of the stone after 3 seconds?

(b) What is the maximal height of the stone?

Solution to 21. The velocity is the derivative of the position (in this problem height). So

$$v(t) = h'(t) = 10 - 2t \quad .$$

The velocity after 3 seconds is $v(3) = 10 - 2 \cdot 3 = 10 - 6 = 4$.

Answer to 21a) 4 m/s

b) When the stone reaches its maximal height its velocity is zero. This is at time t where t solves $v(t) = 10 - 2t = 0$, i.e., $t = 5$. Thus, the solution is $h(5) = 10 \cdot 5 - 5^2 = 50 - 25 = 25$.

Answer to 21b) 25 m

22. Find the absolute maximum and the absolute minimum values of $f(x) = \frac{x}{x^2+1}$ in $[0, 2]$.

Solution to 22. Using the quotient rule, we have

$$f'(x) = \frac{x'(x^2+1) - x(x^2+1)'}{(x^2+1)^2} = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \quad .$$

We have to solve $f'(x) = 0$. Setting the **numerator** equal to 0, we get $1-x^2 = 0$, so $(1-x)(1+x) = 0$ and we get the two roots $x = -1$ and $x = 1$. But $x = -1$ does not satisfy the **residency requirement**, i.e. it does not belong to the interval $[0, 2]$, so we have the three **finalists**: $0, 1, 2$. (The endpoints of the interval always participate in the **final contest**).

Now let's plug these numbers in $f(x) = \frac{x}{x^2+1}$:

$$f(0) = \frac{0}{0^2+1} = 0 \qquad f(1) = \frac{1}{1^2+1} = \frac{1}{2} = .5 \qquad f(2) = \frac{2}{2^2+1} = \frac{2}{5} = .4 \quad .$$

Answer to 22. Absolute minimum value is 0, Absolute maximum value is 1/2.

23. Find the most general antiderivative of the functions

$$a) \quad f(x) = 1 - x^3 + 5x^5 - 3x^7 \quad b) \quad g(x) = \frac{5 - 4x^3 + 3x^6}{x^6}$$

$$c) \quad h(x) = 3e^x + 7\sec^2 x + 5(1 - x^2)^{-\frac{1}{2}}.$$

Solution to 23. Recall the formula

$$\int x^n = \frac{x^{n+1}}{n+1} + C \quad .$$

a)

$$\int f(x)dx = \int (1 - x^3 + 5x^5 - 3x^7)dx = x - \frac{x^4}{4} + 5\frac{x^6}{6} - 3\frac{x^8}{8} = x - (1/4)x^4 + (5/6)x^6 - (3/8)x^8 + C \quad .$$

Answer to 23a) $x - (1/4)x^4 + (5/6)x^6 - (3/8)x^8 + C$

b) First **break it up**, then integrate **term by term**.

$$\int g(x)dx = \int \frac{5 - 4x^3 + 3x^6}{x^6} dx = \int (5 \cdot x^{-6} - 4x^{-3} + 3)dx = 5x^{-5}/(-5) - 4(x^{-2}/(-2)) + 3x =$$

$$-x^{-5} + 2x^{-2} + 3x = \frac{-1}{x^5} + \frac{2}{x^2} + 3x + C \quad .$$

Answer to 23b) $\frac{-1}{x^5} + \frac{2}{x^2} + 3x + C$

c) Recall (or look up in the formula sheet) that $(\tan x)' = \sec^2 x$, hence the **antiderivative** of $\sec^2 x$ is $\tan x + C$. Also $(\sin^{-1} x)' = (1 - x^2)^{-\frac{1}{2}}$, hence the antiderivative of $(1 - x^2)^{-\frac{1}{2}}$ is $\sin^{-1} x + C$. Of course the antiderivative of e^x is $e^x + C$. So

$$\int f(x)dx = \int (3e^x + 7\sec^2 x + 5(1 - x^2)^{-\frac{1}{2}})dx = 3e^x + 7\tan x + 5\sin^{-1} x + C \quad .$$

Answer to 23c) $3e^x + 7\tan x + 5\sin^{-1} x + C$

24. Evaluate the integral if it exists.

$$a) \quad \int_1^9 \frac{\sqrt{u} - 2u^2}{u} du. \quad b) \quad \int_0^2 y^2 \sqrt{1 + y^3} dy. \quad c) \quad \int_1^5 \frac{dt}{(t-4)^2} \quad .$$

$$d) \quad \int_0^1 t^2 \cos(t^3) dt. \quad e) \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx.$$

Solution to 24.

a)

$$\int_1^9 \frac{\sqrt{u}-2u^2}{u}du = \int_1^9 (u^{-\frac{1}{2}} - 2u)du = u^{\frac{1}{2}}/(\frac{1}{2}) - \frac{2u^2}{2}|_1^9 = 2\sqrt{u} - u^2\Big|_1^9 =$$

$$[2\sqrt{9} - 9^2] - [2\sqrt{1} - 1^2] = -76 \quad .$$

Answer to 24a) -76

b) We make the substitution $u = 1 + y^3$ which entails $du/dy = 3y^2$ hence $dy = du/(3y^2)$. Also the limits of the new integral must be computed. When $y = 0$, $u = 1$, and when $y = 2$, $u = 9$. So

$$b)\int_0^2 y^2\sqrt{1+y^3}dy = \int_1^9 y^2\sqrt{u}\frac{du}{3y^2} = \frac{1}{3}\int_1^9 \sqrt{u}du = \frac{1}{3}\int_1^9 u^{\frac{1}{2}}du = \frac{1}{3}\frac{1}{3/2}u^{\frac{3}{2}}|_1^9 =$$

$$\frac{2}{9}\sqrt{u}^3|_1^9 = \frac{2}{9}(\sqrt{9}^3 - \sqrt{1}^3) = \frac{2}{9}(3^3 - 1^3) = \frac{52}{9} \quad .$$

Answer to 24b) $52/9$

c) Beware! The integrand blows up (and hence is undefined) at $t = 4$, that lies **within** the interval of integration.

Answer to 24c Undefined!

d) Here the substitution is $u = t^3$ giving $du/dt = 3t^2$, and so $dt = du/(3t^2)$. As for the limits of integration, when $t = 0$ $u = 0$ and when $t = 1$, $u = 1$.

$$\int_0^1 t^2\cos(t^3)dt = \int_0^1 t^2\cos(u)\frac{du}{3t^2} = \frac{1}{3}\int_0^1 \cos(u)du = \frac{1}{3}\sin u|_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = (\sin 1)/3 \quad .$$

Answer to 24d) $(\sin 1)/3$

e) This is an *indefinite integral* since there are no limits of integration. The answer should be an **expression in** x. In the method of **substitution**, we must translate back at the end.

Take $u = \sqrt{x} = x^{\frac{1}{2}}$, so $du/dx = (1/2)x^{-\frac{1}{2}}$ and $dx = 2x^{\frac{1}{2}}du = 2\sqrt{x}du$. We have

$$e)\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int \frac{e^u}{\sqrt{x}}2\sqrt{x}du = 2\int e^u du = 2e^u = 2e^{\sqrt{x}} + C \quad .$$

Answer to 24e) $2e^{\sqrt{x}} + C$

25. If $\int_1^5 f(x)dx = 12$ and $\int_4^5 f(x)dx = 3.6$, find $\int_1^4 f(x)$.

Solution to 25. We use the obvious relation

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad ,$$

or rather its companion

$$\int_a^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx \quad .$$

So

$$\int_1^4 f(x)dx = \int_1^5 f(x)dx - \int_4^5 f(x)dx = 12 - 3.6 = 8.4 \quad .$$

Answer to 25. 8.4

26. Find the area bounded by the two curves (a) $y_1 = 2x^2$ and $y_2 = 8x$. (b) $y_1 = \sin x$ and $y_2 = \cos x$, $0 \le x \le \frac{\pi}{2}$

Solution to 26. Recall that the area is (if the two curves meet only at two points)

$$\int_a^b (TOP - BOTTOM)dx \quad ,$$

where a and b are the two roots of $y_1 = y_2$. Solving $2x^2 = 8x$ yields $2x(x-4) = 0$ so $x = 0$ and $x = 4$, and indeed there are only two meeting points. To find who's on top just plug-in any x in the interval $(0, 4)$. $x = 1$ is a good choice. When $x = 1$, $y_1 = 2$ and $y_2 = 8$, since $2 < 8$ we know that $y_2 = 8x$ is on top and $y_1 = 2x^2$ is at the bottom. So the area is

$$\int_0^4 (8x - 2x^2)dx = 4x^2 - (2/3)x^3|_0^4 = 4 \cdot 4^2 - (2/3) \cdot 4^3 = \frac{64}{3} \quad .$$

Answer to 26a) $\frac{64}{3}$

b) The curves $y_1 = \sin x$ and $y_2 = \cos x$ meet when $\sin x = \cos x$, and that happens when $x = \pi/4$. There are two parts (equal by symmetry). The one bounded by $y = \sin x$, $y = \cos x$, and the y-axis (alias $x = 0$), and the one bounded by $y = \sin x$, $y = \cos x$, and $x = \pi/2$. The first one happens between $x = 0$ and $x = \pi/4$ and the second between $x = \pi/4$ and $x = \pi/2$. In the first half $y = \cos x$ is on top and $y = \sin x$ is at the bottom, and at the second half it is the other way. Hence the area is

$$\int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx = (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{\pi/2} =$$

$$(\sin(\pi/4) + \cos(\pi/4)) - (\sin(0) + \cos(0)) + (-\cos(\pi/2) - \sin(\pi/2)) - (-\cos(\pi/4) - \sin(\pi/4)) = 2\sqrt{2} - 2.$$

Answer to 26b) $2\sqrt{2} - 2$

27. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin(\frac{i\pi}{n})\frac{\pi}{n} \quad .$$

This is an instance of the formal definition of the **definite integral**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)h \quad ,$$

where $h = (b-a)/n$ and $x_i = a + ih$, $i = 1, 2, \ldots, n$.

The name of the game is to first decide on the function $f(x)$, which in this problem is obviously $f(x) = \sin x$. The argument of sin is $i\pi/n$ so $x_i = i\pi/n$. To get a we plug-in $i = 0$ getting $a = 0$, and to get b we plug in $i = n$, getting $b = \pi$. Hence $h = \pi/n$, which is exactly as in the problem. Hence this complicated-looking limit is nothing but

$$\int_0^{\pi} \sin x = -\cos x|_0^{\pi} = -\cos(\pi) - (-cos(0)) = -(-1) + 1 = 2 \quad .$$

Answer to 27. 2

28. a) Use the definition of the natural logarithm as an integral and compare areas to prove that $\ln 2 < 1 < \ln 3$.

b) Use a) to deduce that $2 < e < 3$.

Solution to 28 a) Recall that

$$\ln x = \int_1^x \frac{1}{t} dt \quad .$$

Hence $\ln 2$ is the area under the curve $y = 1/t$ between $t = 1$ and $t = 2$. To get an **upper bound** for $\ln 2$ note that it is enclosed in the square bounded by the t-axis, the horizontal line $t = 1$, and the vertical lines $t = 1$ and $t = 2$. Its area is $1 \cdot 1 = 1$, so $\ln 2 < 1$.

The **lower bound** $1 < \ln 3$ is more tedious. Take the inscribed rectangles with $n = 8$, getting that $\ln 3$, the area under the curve $y = 1/t$ between $t = 1$ and $t = 3$ is definitely more than

$$\frac{1}{4}\left(\frac{1}{5/4} + \frac{1}{6/4} + \frac{1}{7/4} + \frac{1}{8/4} + \frac{1}{9/4} + \frac{1}{10/4} + \frac{1}{11/4} + \frac{1}{12/4}\right) = \frac{28271}{27720} > 1 \quad .$$

b) Since e^x is an increasing function we can **exponentiate** the inequality $\ln 2 < 1 < \ln 3$ getting

$$e^{\ln 2} < e^1 < e^{\ln 3} \quad .$$

But $e^{\ln w} = w$, so

$$2 < e < 3 \quad .$$