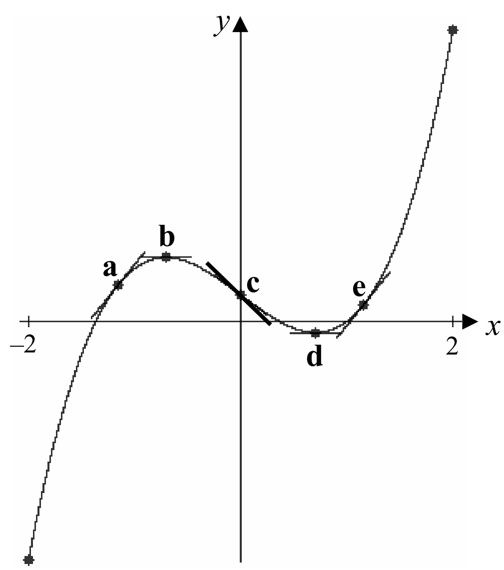
Test 3

Calculus of One Variable I Version **A KEY**

Multiple Choice. There are 19 multiple choice questions. Each question is worth either 3 or 4 points and has one correct answer. The multiple choice problems will count 60% of the total grade. Use a <u>number 2 pencil</u> and bubble in the letter of your response on the scantron sheet for problems 1 – 19. For your own record, also circle your choice on your test since the scantron will not be returned to you. Only the responses recorded on your scantron sheet will be graded. You are NOT permitted to use a calculator on any portion of this test.

1. **(3 pts)** By visual inspection, locate all points on the graph that are guaranteed to exist by the Mean Value Theorem on the interval $[-2, 2]$.



A) Points **b** and **d**

B) Points **a** and **e**

C) Point **c**

D) Points **a**, **c**, and **e**

2. **(3 pts)** Evaluate $\lim_{x\to\infty} \frac{4x+6}{6x^2+3x-2}$.

A) 1

B) 0

C) $\frac{2}{3}$

D) $\frac{1}{3}$

3. **(3 pts)** Assume that the trunk of a tree has a circular cross section. Approximate the change in the radius of the tree when its circumference increases by 2 inches.

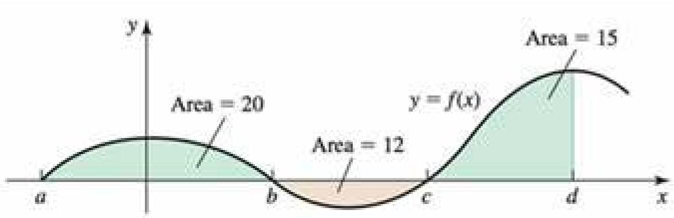
A) $\frac{2}{\pi}$ in.

B) $\frac{\pi}{2}$ in.

C) $\frac{1}{\pi}$ in.

D) π in.

4. **(3 pts)** The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the integral $\int_a^d f(x)\,dx$.



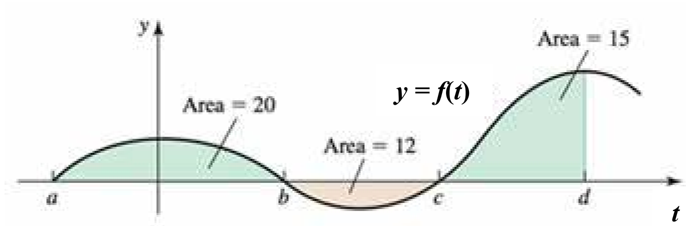
A) 23

B) 47

C) 37

D) 0

5. **(3 pts)** The graph of f is shown in the figure. Let $A(x) = \int_a^x f(t)\, dt$ be an area function for f. Evaluate $A(c)$.



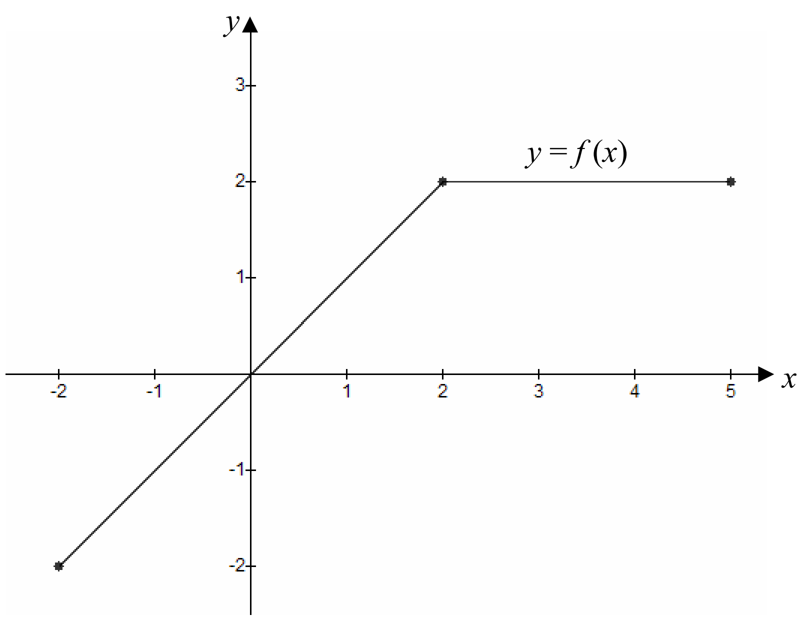
A) 32

B) 8

C) –2

D) –8

6. **(3 pts)** The graph of a piecewise-linear function f, for $-2 \le x \le 5$, is given below. What is the value of $\int_{-2}^{5} f(x)\, dx$?



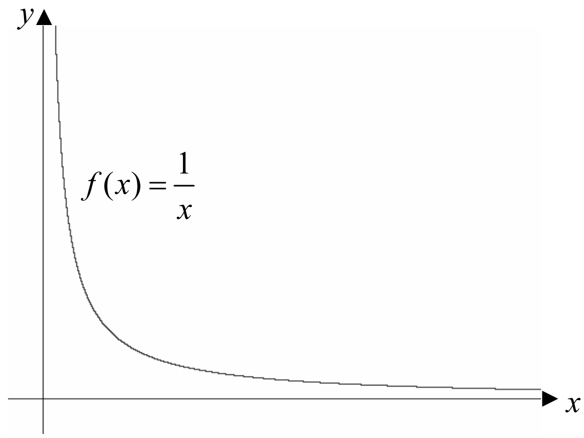
A) 6

B) 8

C) 3

D) 10

7. **(3 pts)** Using a **midpoint** Riemann sum, approximate the area of the region bounded by the graph of $f(x) = \dfrac{1}{x}$ and the x-axis on the interval $[3,\ 7]$ with $n = 2$ subintervals.



A) $\dfrac{2}{5}$

B) $\dfrac{5}{12}$

C) $\dfrac{20}{21}$

D) $\dfrac{5}{6}$

8. **(3 pts)** Determine if Rolle's Theorem applies to the function $f(x) = x^4 - 2x^2$ on the interval $\left[-2,\ 2\right]$. If so, find the point(s) (x-values only) that are guaranteed to exist by Rolle's Theorem.

A) Yes; $x = -1,\ 0,\ 1$

B) No; Rolle's Theorem does not apply.

C) Yes; $x = -\sqrt{2},\ 0,\ \sqrt{2}$

D) Yes; $x = -1,\ 1$

9. **(3 pts)** Suppose $\int_{-2}^{5} h(x)\,dx = 2$ and $\int_{5}^{6} h(x)\,dx = -8$. Evaluate $\int_{-2}^{6} 2h(x)\,dx$.

A) –12

B) 20

C) –3

D) –6

10. **(3 pts)** Evaluate $\sum_{k=1}^{5}\left(k^2 - 7\right)$.

Sum Formulas: $\sum_{k=1}^{n} c = cn$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

A) 48

B) 20

C) 55

D) 18

11. **(3 pts)** Find the linear approximation $L(x)$ to $f(x) = \tan x$ at $a = \pi$.

A) $L(x) = x - \pi$

B) $L(x) = x + \pi$

C) $L(x) = 4x - \pi$

D) $L(x) = x - 4\pi$

12. **(3 pts)** Express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)\,dx$ for the function $y = f(x) = 6x^2 + 5x + 1$.

A) $dy = 12x + 5$

B) $dy = 12x + 5\ dx$

C) $dy = (12x + 5)\ dx$

D) $\dfrac{dy}{dx} = 12x + 5$

13. **(4 pts)** The velocities (in in/sec) of a remote controlled race car moving along a dirt path for 8 seconds are given in the following table. Using n = 4 subintervals, estimate the distance traveled by the car using a **left** Riemann sum approximation on [0, 8].

***t* (sec)**	0	1	2	3	4	5	6	7	8
***v* (in/sec)**	0	10	21	17	27	30	32	12	5

A) 160 in

B) 154 in

C) 170 in

D) 80 in

14. **(3 pts)** Evaluate $\dfrac{d}{dt}\left[\int_0^{\sin t} \left(\dfrac{1}{25-u^2}\right) du\right]$.

A) $\dfrac{1}{25-\sin^2 t}$

B) $\dfrac{\cos t}{25-\sin^2 t}$

C) $\dfrac{1}{\cos t\left(25-\sin^2 t\right)}$

D) $\dfrac{-\cos t}{25-\sin^2 t}$

15. **(3 pts)** Find an antiderivative, $F(x)$, of $f(x)=\dfrac{-27}{x^4}$.

A) $F(x)=\dfrac{9}{x^3}$

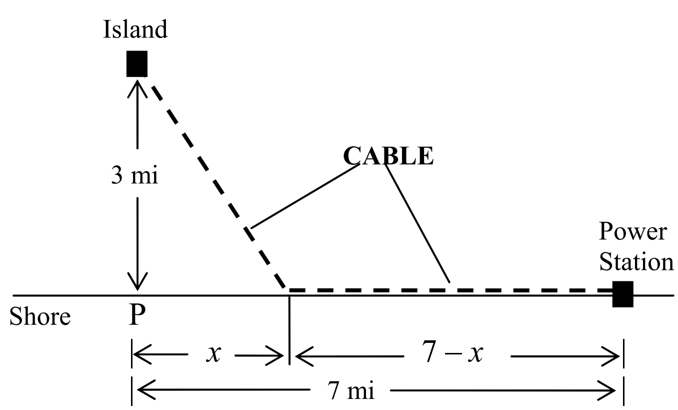
B) $F(x)=\dfrac{-3}{x^9}$

C) $F(x)=\dfrac{3}{x^{10}}$

D) $F(x)=\dfrac{9}{x^4}$

16. **(4 pts)** An island is 3 mi from the nearest point P on a straight shoreline; that point P is 7 mi from a power station (see figure). A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. The cost of running the cable underwater is \$30,000/mi and the cost of running the cable underground is \$10,000/mi. Consider the problem of minimizing the cost of the project by determining the point at which the cable meets the shore. Which equation should be optimized to efficiently solve this problem?

Let x = distance from P to point where underwater cable meets the shore (mi)
C = cost of the project (**thousands of dollars**)



A) $C(x)=10x+30(7-x)$

B) $C(x)=10\sqrt{x^2+9}+30(7-x)$

C) $C(x)=30\left(x^2+9\right)+10(7-x)$

D) $C(x)=30\sqrt{x^2+9}+10(7-x)$

17. **(3 pts)** Evaluate $\int_0^1 \left(1-x^2\right)dx$.

A) $\frac{2}{3}$

B) $\frac{-2}{3}$

C) 1

D) $\frac{-1}{3}$

18. **(3 pts)** Express the sum using sigma notation.

$$4 + 8 + 12 + 16 + 20 + 24$$

A) $\sum_{k=0}^{6} 4(k+1)$

B) $\sum_{k=1}^{6} 4(k+1)$

C) $\sum_{k=1}^{6} 4k$

D) $\sum_{k=2}^{6} 4(k-1)$

19. **(4 pts)** Given the acceleration, $a(t) = 16\cos(4t)$, initial velocity, $v(0) = 2$, and initial position, $s(0) = 4$, of a body moving along a coordinate line at time t, find the body's position at time t.

A) $s(t) = -\cos(4t) + 2t + 4$

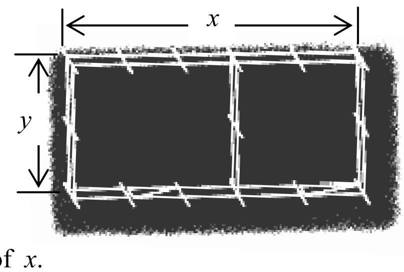
B) $s(t) = -\sin(4t) + 2t + 4$

C) $s(t) = \cos(4t) - 2t + 5$

D) $s(t) = -\cos(4t) + 2t + 5$

Free Response. The Free Response questions will count 39% of the total grade. Read each question carefully. In order to receive full credit you must show legible and logical (relevant) justification which supports your final answer. Give answers as exact answers. You are NOT permitted to use a calculator on any portion of this test.

1. **(10 pts.)** A rancher has 400 ft of fencing with which to enclose two adjacent rectangular corrals (see figure). What are the overall dimensions (x and y) if the area enclosed is to be as large as possible and all of the fencing is used?



Let A = total area enclosed by the fencing (ft^2)
x = length of the enclosed region (ft)
y = width of the enclosed region (ft)

Instructions:
- State the function to be optimized in terms of x.
- State the domain.
- Answer the question including units.

Maximize $A(x) = xy$
subject to $400 = 2x + 3y$
$x, y \geq 0$

So, $y = \dfrac{400-2x}{3}$. We know $x \geq 0$, but we also know $y = \dfrac{400-2x}{3} \geq 0 \rightarrow x \leq 200$.

Maximize $A(x) = x\left(\dfrac{400-2x}{3}\right) = \dfrac{400}{3}x - \dfrac{2}{3}x^2,\ 0 \leq x \leq 200$. **(Open interval also acceptable.)**

$A'(x) = \dfrac{400}{3} - \dfrac{4}{3}x$

$A'(x) = 0$ when $x = \dfrac{400}{3} \cdot \dfrac{3}{4} = 100$ $\qquad$ $A'(x)$ always exists

If the domain is given as a closed interval, then closed interval method may be used.

x	$A(x)$
0	0
100	100(200/3) <= abs. max.
200	0

OR

A' + 0 −
0 100 200 x

OR

$A''(x) = -4/3$
$A''(x) < 0$
for all x in domain of A

So, A has an absolute maximum at x = 100 ft.

When $x = 100$ ft, $y = \dfrac{400-2(100)}{3} = \dfrac{200}{3}$ ft.

The largest area is obtained when $x = 100$ ft and $y = 200/3$ ft.

1. **(continued)** A rancher has 400 ft of fencing with which to enclose two adjacent rectangular corrals (see figure). What are the overall dimensions (x and y) if the area enclosed is to be as large as possible and all of the fencing is used?

Work on Problem:	**Points**
Uses total amount of fence to solve for y in terms of x. (½ point for total amount of fence equation, ½ point for solving for y in terms of x) (May be stated explicitly or stated implicitly in the objective function.)	1 point
Explicitly or implicitly states that $A = xy$.	1 point
Substitutes for y in A to state objective function A in terms of only one variable x.	1 point
States the domain of the objective function. (Interval may be open or closed.)	1 point
Takes the derivative of the objective function.	2 points
Sets the derivative equal to zero. (This must be explicitly stated.)	1 point
Finds the critical point.	1 point
Verifies the location of the absolute maximum through any correct method. (Closed Interval Method may be used only if the domain is given as a closed interval.)	1 point
States answer with appropriate units. (Sentence is not required.)	1 point

Notes:

- Subtract ½ point for not following instructions by stating A in terms of y instead of x.
- Subtract ½ point for missing upper bound on domain.
- No deduction for not explicitly stating that the derivative is always defined (or always exists).
- Subtract ½ point for not explicitly setting the derivative equal to zero.
- Subtract ½ point for using the Closed Interval Method if the domain is not given as closed.
- Subtract ½ point for missing dimension in answer.
- Subtract ½ point for missing or incorrect units.
- Award no points for the final answer if a final answer is stated with no supporting work or egregiously incorrect supporting work.
- Follow errors through for partial credit depending on the severity of the original error, but do not award the 1 point for the correct final answer.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect derivative notation such as: $\frac{d}{dx} = \ldots$ without function inside derivative operator.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

2. **(8 pts.)** Use L'Hôpital's Rule to evaluate the limits. Use of L'Hôpital's Rule must be indicated each time it is used, either symbolically or in words.
No credit will be awarded for any answer given without supporting work.

a. **(4 pts.)** $\lim_{x\to 0}\left[\frac{\cos(5x)-1}{x^2}\right]$

$$\lim_{x\to 0}\left[\frac{\cos(5x)-1}{x^2}\right]$$
$$\underset{(0/0\ i.f.)}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{-5\sin(5x)}{2x}\right)$$
$$\underset{(0/0\ i.f.)}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{-25\cos(5x)}{2}\right)$$
$$=\frac{-25\cos 0}{2}=\frac{-25\cdot 1}{2}$$
$$=\frac{-25}{2}$$

Work on Problem:	Points
Applies L'Hopital's Rule to given limit. (Indeterminate form does not have to be stated.) (½ point for derivative of numerator and ½ point for derivative of denominator)	1 point
Applies L'Hopital's Rule to resulting limit. (Indeterminate form does not have to be stated.) (½ point for derivative of numerator and ½ point for derivative of denominator)	1 point
Evaluates the resulting limit.	1 point
Indicates L'Hopital's Rule each time it is applied. (½ point per application)	1 point

Notes:
- Subtract 4 points for irrelevant work such as using the quotient rule to take the derivative.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, missing or incorrect limit notation such as: $\lim_{x\to 0} = \ldots$ without function inside limit, inappropriate limit notation: $\underset{(0/0\ i.f.)}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{-25\cos(5x)}{2}\right)=\lim_{x\to 0}\left(\frac{-25}{2}\right)$, and the untrue statement: $\lim_{x\to 0}\left[\frac{\cos(5x)-1}{x^2}\right]=\frac{0}{0}$.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

2.**(continued)** Use L'Hôpital's Rule to evaluate the limits. Use of L'Hôpital's Rule must be indicated each time it is used, either symbolically or in words.
No credit will be awarded for any answer given without supporting work.

b. **(4 pts.)** $\lim\limits_{\theta\to 0^+}(2\theta\cot 3\theta)$

$$\lim_{\theta\to 0^+}(2\theta\cot 3\theta) \qquad 0\cdot\infty \text{ i.f.}$$

$$= \lim_{\theta\to 0^+}\left(\frac{2\theta\cos 3\theta}{\sin 3\theta}\right)$$

$$\underset{(0/0\ i.f.)}{\overset{L}{=}} \lim_{\theta\to 0^+}\left(\frac{[2]\cos 3\theta + 2\theta[-3\sin 3\theta]}{3\cos 3\theta}\right)$$

$$= \lim_{\theta\to 0^+}\left(\frac{2\cos 3\theta - 6\theta\sin 3\theta]}{3\cos 3\theta}\right)$$

$$= \frac{2\cos 0 - 6(0)\sin 0}{3\cos 0} = \frac{2\cdot 1 - 0}{3\cdot 1}$$

$$= \frac{2}{3}$$

Work on Problem:	Points
Recognizes implicitly or explicitly that the original limit is an indeterminate form and algebraically manipulates the limit to re-express as indeterminate form 0/0.	1 point
Applies L'Hopital's Rule to resulting limit. (Indeterminate form does not have to be stated.) (1 point for derivative of numerator and ½ point for derivative of denominator)	1.5 points
Evaluates the resulting limit.	1 point
Indicates L'Hopital's Rule each time it is applied.	½ point

Notes:
- Subtract 4 points for irrelevant work such as using the quotient rule to take the derivative.
- Subtract ½ point for each incorrect trigonometric identity.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals including "$= 0\cdot\infty$", missing or incorrect use of parentheses, missing or incorrect limit notation such as: $\lim\limits_{\theta\to 0^+} = \ldots$ without function inside limit, and inappropriate limit notation.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

3. **(9 pts.)** Evaluate the following integrals.

a. **(4 pts.)** $\displaystyle\int\left(\frac{\sqrt{y}}{7}+\frac{8}{\sqrt{y}}-1\right)dy$

$$=\int\left(\frac{1}{7}y^{1/2}+8y^{-1/2}-1\right)dy \qquad =\frac{1}{7}\cdot\frac{y^{3/2}}{3/2}+8\left(\frac{y^{1/2}}{1/2}\right)-y+C$$

$$\left(=\frac{1}{7}\cdot\frac{2}{3}y^{3/2}+8\cdot 2y^{1/2}-y+C=\frac{2}{21}y^{3/2}+16y^{1/2}-y+C\right)$$

Work on Problem:	Points
Takes the antiderivative of each term. (1 point per term)	3 points
Adds constant of integration: "+ C."	1 point

Notes:
- Simplification is not required.
- Subtract ½ point for each incorrect trigonometric identity.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect integral notation.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

b. **(5 pts.)** $\displaystyle\int_{\pi/2}^{3\pi/2}(6\cos x)\ dx$

$$=\left[6\sin x\right]_{\pi/2}^{3\pi/2}=6\sin\left(\frac{3\pi}{2}\right)-6\sin\left(\frac{\pi}{2}\right)=6(-1)-6(1)=-12$$

Work on Problem:	Points
Takes the antiderivative of the integrand.	1 point
Substitutes the upper and lower limits into the result. (1 point per evaluation)	2 points
Subtracts the value of the antiderivative at the lower limit from the antiderivative at the upper limit. (Applies the Fundamental Theorem of Calculus.)	1 point
Evaluates the result.	1 point

Notes:
- Subtract ½ point maximum for not evaluating the trigonometric function or evaluating incorrectly.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect integral notation.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

4. **(12 pts.)** Compute the definite integral $\int_0^1 (1-x^2)\,dx$ as the limit of a **right** Riemann sum.

a. **(1 pt.)** Let n be the number of subintervals into which the interval $[0, 1]$ is to be divided. State Δx, the width of each subinterval, in terms of n.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

Work on Problem:	**Points**
States Δx.	1 point
Notes: • Graded all or nothing.	

b. **(1 pt.)** Find an expression for the **right endpoint** of the k-th subinterval, x_k.

$$x_k = 0 + k\frac{1}{n} = \frac{k}{n}$$

Work on Problem:	**Points**
States x_k .	1 point
Notes: • Graded all or nothing. Simplification not required.	

c. **(4 pts.)** State the **right** Riemann sum for $f(x) = 1 - x^2$ on the interval $[0, 1]$.

$$f(x_k) = f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2 = 1 - \frac{k^2}{n^2}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(1 - \frac{k^2}{n^2}\right)\frac{1}{n}$$

Work on Problem:	**Points**
Substitutes x_k into integrand $f(x)$. (Neither separate step nor simplification required. But, function should be evaluated, not left as $f(x_k)$ or $f\left(\frac{k}{n}\right)$.)	2 points
Sets up the right Riemann sum.	2 points
Notes: • Subtract 1 point for not making the substitution for x_k in terms of k and n. • Award full credit for substituting result of part b into integrand, even if part b was incorrect. • Award full credit for multiplying result of $f(x_k)$ by result of part a, even if parts were incorrect. • Subtract 1 point for each incorrect substitution into the Riemann sum. That is, work does not follow. • Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect summation notation. • Subtract ½ point for each minor algebra, arithmetic, and/or copy error.	

4.**(continued)** Compute the definite integral $\int_0^1 (1-x^2)\,dx$ as the limit of a **right** Riemann sum.

d. **(4 pts.)** Simplify the Riemann sum from part c using the sum formulas.
State your final answer in terms of ***n*** only.

$$\sum_{k=1}^{n} c = cn, \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2+n}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{k=1}^{n}\left(1-\frac{k^2}{n^2}\right)\frac{1}{n} = \frac{1}{n}\left[\sum_{k=1}^{n} 1 - \frac{1}{n^2}\sum_{k=1}^{n} k^2\right]$$
$$= \frac{1}{n}\left[n - \frac{1}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}\right] = \frac{1}{n}\left[n - \frac{2n^2+3n+1}{6n}\right] = \frac{1}{n}\left[n - \frac{n}{3} - \frac{1}{2} - \frac{1}{6n}\right]$$
$$= \frac{1}{n}\left[\frac{2n}{3} - \frac{1}{2} - \frac{1}{6n}\right] = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

Work on Problem:	Points
Uses the algebraic properties to distribute the sum.	1 point
Evaluates the summations using the constant sum rule and a sum formula.	2 points
Simplifies the result.	1 point

Notes:
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect summation notation.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

e. **(2 pts.)** Find the exact value of the definite integral $\int_0^1 (1-x^2)\,dx$ by taking a limit.

$$\int_0^1 (1-x^2)\,dx = \lim_{n\to\infty}\left(\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right) = \frac{2}{3} - 0 - 0 = \frac{2}{3}$$

Work on Problem:	Points
Sets up the limit as $n \to \infty$.	1 point
Evaluates the limit.	1 point

Notes:
- Award full credit if work follows from part d.
- Simplification is not required to take the limit.
- Restatement of the definite integral is not required.
- Award full credit for any correct method of taking the limit. Students may state the answer for this limit without showing work.
- Subtract 1 point for not showing the statement of the limit of the result from part d.
- Subtract ½ point for each type of notation error with a maximum of 1 point deduction for all notation errors. Types of notation errors include, but are not limited to, missing or incorrect use of equals, missing or incorrect use of parentheses, and missing or incorrect limit notation.
- Subtract ½ point for each minor algebra, arithmetic, and/or copy error.

Scantron **(1 pt.)**

Check to make sure your Scantron form meets the following criteria. If any of the items are NOT satisfied when your Scantron is handed in and/or when your Scantron is processed one point will be subtracted from your test total.

My Scantron:

- ☐ is bubbled with firm marks so that the form can be machine read;
- ☐ is not damaged and has no stray marks (the form can be machine read);
- ☐ has **19** bubbled in answers;
- ☐ has **MthSc 106** and my Section number written at the top;
- ☐ has my Instructor's name written at the top;
- ☐ has Test No. **3** written at the top;
- ☐ has Test Version **A** both written at the top and bubbled in below my CUID;
- ☐ and shows my correct CUID both written and bubbled in.